which is known as the Landau criterion for superflow. The velocity \( v_c \) is called the critical velocity of superflow; it marks an "upper limit" to the flow velocities at which the fluid exhibits superfluid behavior. The observed magnitude of the critical velocity varies significantly with the geometry of the channel employed; as a rule, the narrower the channel the larger the critical velocity. The observed values of \( v_c \) range from about 0.1 cm/sec to about 70 cm/sec.

The theoretical estimates of \( v_c \) are clearly of interest. On one hand, we find that if the excitations obey the ideal-gas relationship, viz. \( \varepsilon = \frac{p^2}{2m} \), then the critical velocity turns out to be exactly zero. Any velocity \( v \) is then greater than the critical velocity; accordingly, no superflow is possible at all. This is a very significant result, for it brings out very clearly the fact that interatomic interactions in the liquid, which give rise to an excitation spectrum different from the one characteristic of the ideal gas, play a fundamental role in bringing about the phenomenon of superfluidity. Thus, while an ideal Bose gas does undergo the phenomenon of Bose–Einstein condensation, it cannot support the phenomenon of superfluidity as such. On the other hand, we find that (i) for phonons, \( v_c = c \approx 2.4 \times 10^9 \) cm/sec and (ii) for rotons, \( v_c = \left( \frac{p_0^2 + 2\mu \Delta}{\mu} - p_0 \right) / \mu \approx \frac{\Delta}{p_0} \approx 6.3 \times 10^3 \) cm/sec, which are too high in comparison with the observed values of \( v_c \). In fact, there is another type of collective excitations which can appear in liquid helium II, viz. quantized vortex rings, with an energy–momentum relationship of the form: \( \varepsilon \propto p^{1/2} \). The critical velocity for the creation of these rings turns out to be numerically consistent with the experimental findings; not only that, the dependence of \( v_c \) on the geometry of the channel can also be understood in terms of the size of the rings created.

For a review of this topic, especially in regard to Feynman's contributions, see Mehra and Pathria (1994); see also Secs 10.4–10.6 of this text.

**Problems**

7.1. By considering the order of magnitude of the occupation numbers \( \langle n_i \rangle \), show that it makes no difference to the final results of Sec. 7.1 if we combine a finite number of \((\epsilon \neq 0)\)-terms of the sum (7.1.2) with the \((\epsilon = 0)\)-part of eqn. (7.1.6) or include them in the integral over \( \varepsilon \).

7.2. Deduce the virial expansion (7.1.13) from eqns (7.1.7) and (7.1.8), and verify the quoted values of the virial coefficients.

7.3. Combining eqns (7.1.24) and (7.1.26), and making use of the first two terms of formula (D. 9), show that, as \( T \) approaches \( T_c \) from above, the parameter \( \alpha(= -ln\varepsilon) \) of the ideal Bose gas assumes the form

\[
\alpha \approx \frac{1}{\pi} \left( \frac{3\pi^2/2}{4} \right) \left( \frac{T - T_c}{T_c} \right)^2.
\]

7.4. Show that for an ideal Bose gas

\[
\frac{1}{\varepsilon} \left( \frac{\partial \varepsilon}{\partial T} \right)_p = -\frac{5}{2T} \frac{g_1/2(z)}{g_1/2(z)}.
\]

cf. eqn. (7.1.36). Hence show that

\[
\gamma \equiv \frac{C_P}{C_V} = \frac{\left( \partial \varepsilon / \partial T \right)_p}{\left( \partial \varepsilon / \partial T \right)_V} = \frac{5}{3} \frac{g_1/2(z)g_1/2(z)}{g_1/2(z)^2},
\]

as in eqn. (7.1.48b). Check that, as \( T \) approaches \( T_c \) from above, both \( \gamma \) and \( C_P \) diverge as \((T - T_c)^{-\gamma}\).
7.5. (a) Show that the isothermal compressibility \( \kappa_T \) and the adiabatic compressibility \( \kappa_S \) of an ideal Bose gas are given by

\[
\kappa_T = \frac{1}{n_k T} \frac{g_{1/2}(z)}{g_{3/2}(z)}, \quad \kappa_S = \frac{3}{5n_k T} \frac{g_{3/2}(z)}{g_{5/2}(z)},
\]

where \( n = N/V \) is the particle density in the gas. Note that, as \( z \to 0 \), \( \kappa_T \) and \( \kappa_S \) approach their respective classical values, viz. \( 1/P \) and \( 1/\rho P \). How do they behave as \( z \to 1 \)?

(b) Making use of the thermodynamic relations

\[
C_P - C_V = T \left( \frac{\partial P}{\partial T} \right)_V \left( \frac{\partial V}{\partial T} \right)_P = TV \kappa_T \left( \frac{\partial P}{\partial T} \right)_V^2
\]

and

\[
\frac{C_P}{C_V} = \frac{\kappa_T}{\kappa_S}
\]

derive eqns (7.1.48a) and (7.1.48b).

7.6. Show that for an ideal Bose gas the temperature derivative of the specific heat \( C_V \) is given by

\[
\frac{1}{N_k} \left( \frac{\partial C_V}{\partial T} \right)_V = \begin{cases} 
1 & \text{for } T > T_c, \\
- \frac{45}{8} \left( \frac{g_{5/2}(z)}{g_{3/2}(z)} \right) - \frac{9}{4} \left( \frac{g_{3/2}(z)}{g_{1/2}(z)} \right) - \frac{27}{8} \left( \frac{g_{5/2}(z)}{g_{3/2}(z)} \right)^2 \frac{g_{-1/2}(z)}{g_{1/2}(z)} & \text{for } T < T_c.
\end{cases}
\]

Using these results and the main term of formula (D.9), verify eqn. (7.1.38).

7.7. Evaluate the quantities \( \left( \frac{\partial^2 P/\partial T^2}{} \right)_V \), \( \left( \partial^2 \mu/\partial T^2 \right)_V \) and \( \left( \partial^2 \mu/\partial T^2 \right)_P \) for an ideal Bose gas and check that your results satisfy the thermodynamic relationships

\[
C_V = VT \left( \frac{\partial^2 P}{\partial T^2} \right)_V - NT \left( \frac{\partial \mu}{\partial T^2} \right)_V
\]

and

\[
C_P = -NT \left( \frac{\partial \mu}{\partial T^2} \right)_P.
\]

Examine the behavior of these quantities as \( T \to T_c \) from above and from below.

7.8. The velocity of sound in a fluid is given by the formula

\[
w = \sqrt{(\partial P/\partial \rho)_T},
\]

where \( \rho \) is the mass density of the fluid. Show that for an ideal Bose gas

\[
w^2 = \frac{5kT}{3m} \frac{g_{5/2}(z)}{g_{3/2}(z)} = \frac{5}{9} \left( \langle u^2 \rangle \right),
\]

where \( \langle u^2 \rangle \) is the mean square speed of the particles in the gas.

7.9. Show that for an ideal Bose gas

\[
\langle u \rangle = \frac{4}{\pi} \frac{g_1(z)g_2(z)}{g_3(z)} \frac{1}{\langle u^2 \rangle},
\]

\( u \) being the speed of a particle. Examine and interpret the limiting cases \( z \to 0 \) and \( z \to 1 \); cf. Problem 6.6.

7.10. Consider an ideal Bose gas in a uniform gravitational field (of acceleration \( g \)). Show that the phenomenon of Bose–Einstein condensation sets in at a temperature \( T_c \) given by

\[
T_c \approx T_c^0 \left[ 1 + \frac{8}{9\xi} \left( \frac{1}{2} \right) \left( \frac{\pi m g L}{kT_c^0} \right)^{1/2} \right].
\]