If $\Delta t > \Delta x$, it is in the state of equilibrium (6).

If $\Delta t < \Delta x$, it is an old equilibrium (4).

Lecture 5 - New equilibrium state - read pages 32-35 in my notes.

HW Section.

Shock Waves and Areaks of Packets

Until now we discussed small perturbations around an equilibrium state. Now, we will discuss situations in which the wave "breaks." Thinks of an example in waves in the ocean - in deep waters, the characteristic velocity $c$ is very large as the fact that the velocity of the particles $u$ velocity of the waves allow us to do a perturbative treatment. But in shallow waters, the characteristic velocity of the wave falls and the velocity of the particles exceeds the wave velocity. The particles will tend to pass the wave, and the wave "breaks." - the energy is channeled to a turbulent motion and finally for heat.

Some things will happen, a sound wave will "break" if the plasma velocity approaches $V_0$. Why?

Suppose we have a wave with amplitude of density $\rho$. If the next losses are negligible, $\alpha = 1$ and

$$\frac{\alpha}{\rho_0} = \frac{2}{\rho_1} \Rightarrow \frac{\rho_0}{\rho_1} = \frac{2}{\rho_0}$$

or

$$\frac{\alpha}{\rho_0} = \left(\frac{\rho_1}{\rho_0}\right)^{2} \Rightarrow \frac{\rho_0}{\rho_1} = \left(\frac{\rho_0}{\rho_1}\right)^{2}$$

$$\frac{\alpha}{\rho_0} = \left(\frac{\rho_1}{\rho_0}\right)^{2} \Rightarrow \frac{\rho_0}{\rho_1} = \left(\frac{\rho_0}{\rho_1}\right)^{2}$$
The sound speed is given by (3.50) as

\[ V_s = \left( \frac{0.5 kT}{m} \right)^{1/2} \]  

(3.2)

so as the wave moves, there is a variation

\[ \frac{\delta V_s}{V_s} = \frac{4}{10} \overset{\dashrightarrow}{\frac{1}{3}} \frac{\delta T}{T_0} = \frac{1}{3} \frac{\delta T}{T_0} \]  

(3.3)

In the limit when \( \delta T \to 0 \) this effect is negligible and all the parts of the wave go with the same velocity \( V_s \). As a result, the wave is still a sinusoid. But if \( \delta T \) is finite, Eq. (3.3) shows that the wave will travel faster when \( \delta T \) is large. The effect on the wave is that it becomes with a higher "pitch" as it moves forward.

We can estimate how long takes the wave to increase substantially

\[ t \approx \frac{4}{3} V_s \]  

(3.4)

using \( \delta T = \frac{4}{3} V_s \eta \). Here \( \eta \) is the period of the wave.

For a linear regime, when \( \rho_1 \to 0 \) the time of growth of the wave \( \to \infty \).

However, if \( \rho_1 \) is finite the growth time can be small.

The wave in the ocean "breaks" when the gravitation acceleration acting on the water down.
In an acoustic wave, described by fluid equations, the variables should conserve mass and momentum. However, macroscopically, particle in the steep portion of the wave moves ahead of the particles transmitted to them, moment by moment, through several mean-free-paths. In the most "steep" part of the wave, we can expect that the particle velocities will present an anomalous distribution resulting from a mixture of slow particles ahead of the "steep" part of the wave and fast particles behind it.

Usually viscosity is negligible — it's not negligible when the flow velocity has large gradients. When we have this situation, with viscosity, the momentum equation is

$$\rho \frac{D}{Dt} \mathbf{v} = -\nabla p - \rho \mathbf{v} \times \mathbf{β} + \frac{1}{c^2} \frac{\partial \mathbf{v}}{\partial t} + \mathbf{F}_v$$

where $\mathbf{β} = \rho \mathbf{v} \times \mathbf{v}$ and

$$\mathbf{F}_v = \mathbf{F}_v \left( \mathbf{v} + \frac{1}{3} \mathbf{v} \left( \mathbf{v} \cdot \mathbf{v} \right) \right)$$

(Landau & Lifshitz, p. 48)

Let's neglect gravitational and magnetic forces and describe the flow in the $x$ axis:

$$\frac{d \mathbf{v}}{dt} = -\frac{\partial p}{\partial x} + \frac{\mathbf{v} \cdot \nabla \mathbf{v}}{3} \frac{1}{\partial x} \left( \frac{\partial \mathbf{v}}{\partial x} \right)$$

$$= -\frac{\partial p}{\partial x} + \frac{4}{3} \frac{\partial \mathbf{v}^2}{\partial x} \rightarrow \mathbf{F}_v \frac{\partial \mathbf{v}^2}{\partial x} < 0 \rightarrow \text{so this will lead to slow down the fluid}$$

$$\mathbf{F}_v \frac{\partial \mathbf{v}^2}{\partial x} \rightarrow x$$

$$\frac{\partial p}{\partial x} \text{ will be to minimize } \mathbf{F}_v > 0 \rightarrow \text{ accelerate the flow.}$$
The ratio between the damping force and force on growth on the peak is:

\[
\frac{4 \pi \frac{dv}{dt}}{3 \frac{dp}{dx}} = \frac{8v}{\rho \frac{d^2x}{dx^2}} = \frac{g}{\rho \bar{x}}
\]

The viscosity is important when the ratio approaches 1, or

\[
x = \frac{g}{\rho}
\]

where \(\rho = \rho_v\) and \(v = \bar{v}_v\) when the wave is sharp.

\[
\frac{x}{\bar{x}} \approx \frac{\rho_v}{\rho_v} \approx \frac{v}{\bar{v}_v}
\]

The Reynolds number corresponds to: (before we had \(\text{Re} = \frac{\bar{V}_A}{\nu}\))

\[
\text{Re} = \frac{\bar{v}_v A}{\nu} \approx 1
\]

\(\nu\) is the diffusion coefficient so

\[
\nu = m \bar{v}_p \nu_v \quad \text{where } \nu_v \text{ mean-free-path}.
\]

So

\[
x \approx m \bar{v}_p
\]

or when the viscosity becomes important when \(x \approx m \bar{v}_p\).

Or when the collisions between particles behind the wave interact with particles ahead of the wave front and this can only happen if the two ensembles of particles are separated only by few mean free paths.

The fluid can tell us when wave will "break" but because the theory is invalid in scales \(m \bar{v}_p\), it cannot describe what happen in detail.
Thanks to, we don't really need to know—yet we just need to determine that a shock front was triggered and that we can use the conservation laws.

The strength of the shock:

$$H = \frac{V_s}{\sqrt{\gamma + 1}} V'_s$$

where $V_s$ is the shock velocity with respect to the flow and $V'_s$ is the sound speed ahead of the shock.

All these ideas apply to "collisional shock" defined as the shock wave by
collisions of particles. In plasmas, we can have collisionless shocks in which
the acceleration happens through inertia wave-particle.

We don't care if it is collisional or collisionless—we really only care about
the general properties of the fluid.

3.2 Rankine-Hugoniot Relations

Let's study a particular case of a shock propagating in a direction $\hat{e}_x$.
The case $B = 0$ can be obtained putting $B = 0$ in the eq. as the motion
along $B$ don't affect $B$.

The appropriate equations are:

$$\frac{\partial \rho}{\partial t} + \frac{\partial (\rho v)}{\partial x} = 0 \quad (2.12) \quad \text{conservation law}$$

and

$$\frac{\partial (\rho v)}{\partial t} + \frac{\partial (\rho v^2 + \frac{1}{2} \rho \gamma T)}{\partial x} = \frac{\partial}{\partial x} \left( \frac{\partial B^2}{\partial t} - \frac{\partial B^2}{\partial t} \hat{e}_x + \frac{1}{\mu_0} \frac{\partial B}{\partial x} \right)$$

$$+ \frac{\partial}{\partial x} \left( \frac{\partial B}{\partial t} - \frac{\partial B}{\partial t} \hat{e}_x + \frac{1}{\mu_0} \frac{\partial B}{\partial x} \right)$$

$$- \frac{\rho v}{\mu_0} \left( \frac{\partial B}{\partial x} \right) = 0 \quad (3.11)$$
where

\[
\begin{align*}
T_{ij} &= \rho \delta_{ij} + \rho v_i v_j - \delta_{ij} = \rho \delta_{ij} + \rho v_i v_j - 3 (2 x_i v_j - 2 \delta_{ij} v_i v_j) \\
T_{ij} &= \frac{B^2 \delta_{ij}}{8 \pi} - 4 B_i B_j 
\end{align*}
\]

(3.16)

Example for pressure page:

solar wind \quad \text{shock} \quad B_0 \rightarrow 0 \Rightarrow \Delta B_0

\Delta x = 10^{13} \text{km} \quad (measured)

\Delta B_0 \approx 10^6 \text{m}

\Delta x \approx 10^8 \text{m}

\text{collisions}

Why \quad \Delta B_0 > \Delta x \quad ??

Because \quad \Delta B_0 \approx \Delta x \quad \text{is not significant} \Rightarrow \text{it's not due to collisions}

Let's assume that the flow is stationary \quad \text{so} \quad \frac{\partial}{\partial t} = 0 \quad \text{Let's neglect}

\text{aerodynamics} \quad \text{let's adopt a reference system at that is at }\text{ rest in}

\text{the shock}

The plasma in region \text{4 and 5} \quad \text{obeys} \quad \text{MHD} \quad \rightarrow \text{in the interface}

\text{we cannot see} \quad \text{MHD}

\text{downstream} \quad \uparrow \quad \text{upstream} \quad \downarrow

\begin{align*}
A_\perp &= \frac{V_y}{v_i} \\
A_\| &= \frac{v_i}{V_y}
\end{align*}

\text{subscript}
\[ v^2 = u^2 \]
\[ \Omega = \frac{8}{3} \dot{\theta} \]
\[ \dot{\theta} \dot{\theta} = \dot{\theta} \]

The only component of \( \tilde{T}_{ij} \) that does not go to zero is
\[ \tilde{T}_{xx} = \rho x^2 - \rho \dot{x}^2 = \rho + \rho \dot{x}^2 - \frac{8}{3} (2 \dot{x}^2 - \frac{2}{3} x^2) \]
\[ \tilde{T}_{xx} = \rho + \rho \dot{x}^2 - \frac{8}{3} \dot{x}^2 \]

Same for \( \tilde{T}_{ij} \): \( \tilde{T}_{xx} = \rho x^2 - \rho \dot{x}^2 = \rho \]
\[ \int_{S} (\rho x^2) \cdot dS = 0 \quad \text{(not zero in \( S_x \) and \( S_\theta \))} \]

\[ \text{in (4.12):} \quad \int_{S} (\rho \dot{x}) \cdot dS = 0 \]

\[ \text{in (4.13):} \quad \int_{S} (\tilde{T}_{xx} + \tilde{T}_{\theta \theta}) \cdot dS = 0 \]

- \( \tilde{T}_{yy} \) does not enter because \( \tilde{T}_{yy} = \tilde{T}_{yy}(s) \).

\[ \int_{S} \tilde{T}_{yy} \cdot dS = 0 \]

\[ = T_{yy}(s) \int_{S} dS = T_{yy}(S_x - S_\theta) = 0 \]
\[ \text{In } \left( 4.14 \right) = \frac{1}{2} \int_{G} \left[ \frac{1}{8} \left( v^2 B^2 + e_v B^2 \right) + \rho \left( \frac{\mu}{v} + \frac{\mu}{B} \right) \right] \left( \frac{\mu}{v} + \frac{\mu}{B} \right) + \left( v \cdot \nabla \right) - \frac{2}{3} \nabla \left( \frac{1}{8} \left( v^2 B^2 + e_v B^2 \right) + \rho \left( \frac{\mu}{v} + \frac{\mu}{B} \right) \right) \cdot \nabla \right) \cdot ds = 0 \]  

\[ (3.21) \]

\[ - \rho V_1 A_1 + \rho V_2 A_2 = 0 \quad \rho V_1 = \rho V_2 \quad (3.22) \]

\[ A_1 = A_2 \]

\[ \rho V_1, \rho V_2 \text{ are obtained long distances from the shock. (To guarantee that the gas is normalized.)} \]

\[ \text{Notation: } \left[ p \rho \right] = \rho V_1 - \rho V_1 \]

\[ \text{so } (3.22) \text{ becomes } \left[ p \rho \right] = 0 \quad (3.23) \]

\[ \left( \frac{\mu}{v} A_1 + \frac{\mu}{B} A_1 \right) + \frac{1}{2} \frac{\mu}{v} A_2 + \frac{1}{2} \frac{\mu}{B} A_2 = 0 \]

\[ \left[ \frac{\mu}{v} A_1 + \frac{\mu}{B} A_1 \right] = 0 \]

\[ \left[ p + \rho v^2 + \frac{\mu}{3} \left( \frac{\mu}{v} \right) \right] = 0 \quad (3.24) \]

From a form of the energy equation:

\[ \frac{\partial B}{\partial t} \cdot \left[ \frac{1}{8} \left( v^2 B^2 + e_v B^2 \right) + \rho \left( \frac{\mu}{v} + \frac{\mu}{B} \right) \right] = 0 \quad (3.25) \]

\[ x \cdot \nabla B = 0 \quad \text{because } B \text{ is a function of } x \]

\[ \frac{\partial B}{\partial t} \left( x \cdot \nabla B \right) = \frac{\partial B}{\partial \xi} \left( x \cdot \nabla B \right) + \frac{\partial B}{\partial \eta} \left( x \cdot \nabla B \right) \quad (3.26) \]

\[ \text{non-zero in } A_1 \text{ or } A_2 \]
\[ p^2 (x) \text{ d}x + 2 \text{ are zero in } A_1 \text{ and } A_2 \]
\[ (\frac{\partial^2}{\partial x^2}) \psi = -\frac{1}{\hbar^2} \left[ \frac{\partial^2 \psi}{\partial x^2} - \frac{4}{3} \frac{\partial \psi}{\partial x} \right] + \left( \frac{3}{8} \frac{\partial^2 \psi}{\partial x^2} \right) = 0 \]
\[ \psi = \psi(x) \text{ (not zero in } A_1 \text{ and } A_2) \]
\[ \psi = \psi(x) \]
\[ \psi(\infty) \]
\[ \psi(0) = \psi(\infty) \]
\[ \psi(x) \text{ satisfies the Schrödinger equation} \]
\[ [\frac{\partial^2}{\partial x^2} - \frac{4}{3} \frac{\partial}{\partial x} + \frac{3}{8} \frac{\partial^2}{\partial x^2}] \psi = 0 \]
\[ \psi(x) = \psi(x) \text{ (not zero in } A_1 \text{ and } A_2) \]
\[ \psi(x) \text{ satisfies the Schrödinger equation} \]
\[ [\frac{\partial^2}{\partial x^2} - \frac{4}{3} \frac{\partial}{\partial x} + \frac{3}{8} \frac{\partial^2}{\partial x^2}] \psi = 0 \]
Using the induction equation:

\[
\frac{\partial B^3}{\partial t} = -\nabla \cdot (v \times B^3) + \nabla \times (\nabla \times \mathbf{B}^3) + \nabla \cdot \mathbf{J}
\]

\[
= 0
\]

\[
-\nabla \cdot (v \times B^3 - B \times \nabla v) + \nabla \times (\nabla \times \mathbf{B}^3) = 0
\]

\[
\partial_t (v \times B^3) + \nabla \times (\nabla \times \mathbf{B}^3) = 0
\]

\[
\int_v (v \times B^3 + \nabla \times \mathbf{B}^3) \cdot ds = 0
\]

\[
[v B^3] = 0 \Rightarrow \left[ \frac{\rho v B^3}{\rho} \right] = 0
\]

\[
\rho = 0 \quad (v)
\]

\[
\begin{align*}
\text{Unknowns:} & \quad B_2, \rho_2, \gamma_2, \Gamma_2 \\
\text{New variables:} & \quad x = \frac{B_2}{\lambda}, \quad \chi = \frac{\Gamma_2}{\mu}, \quad \gamma = \frac{\rho_2}{\rho}, \quad \gamma = \frac{\Gamma_2}{\mu} \\
\rightarrow & \quad \beta = \left( \frac{B_2^2}{\lambda} \right) / \mu
\end{align*}
\]

\[
\frac{M}{\varepsilon} = \frac{v_4 \varepsilon V_{\gamma}}{x^2 v_4 \varepsilon} = \frac{v_4 \varepsilon V_{\gamma}}{1 + \frac{8 \rho_2 \varepsilon}{\mu}}
\]

\[
\text{From Eq. (v):} \quad B_2 = \frac{\rho_2}{\rho}
\]
From Eq. (1) \( V_a = p_a V_a / \rho_a \)

From Eq. (2) and from the above equations:

\[ p - p_a = \rho_a \frac{\beta_a}{\rho_a} \frac{\beta_a}{\rho_a} = \rho_a \frac{\beta_a}{\rho_a} \frac{\beta_a}{\rho_a} = \rho_a \frac{\beta_a}{\rho_a} \frac{\beta_a}{\rho_a} = \frac{p - p_a}{\gamma} = \frac{p - p_a}{\gamma} \]

\[ \rho_a V_a^2 \left( \frac{1}{\gamma} - 1 \right) = \frac{B_a}{\gamma} \left( \frac{1}{\gamma} - 1 \right) + p - p_a \] (5)

From Eq. (3) \( \frac{B_a}{\gamma} + \frac{V_a^2}{\gamma - 1} \left( \frac{p - p_a}{\rho_a} + \frac{p_a B_a^2}{\gamma - 1} \left( \frac{1}{\gamma} - 1 \right) \right) \)

\[ \sqrt{\frac{p - p_a}{\gamma - 1}} \left( \frac{1}{\gamma} - 1 \right) = \frac{B_a}{\gamma} \left( \frac{1}{\gamma} - 1 \right) + p - p_a \]

\[ \sqrt{\frac{p - p_a}{\gamma - 1}} \left( \frac{1}{\gamma} - 1 \right) = \frac{B_a}{\gamma} \left( \frac{1}{\gamma} - 1 \right) + p - p_a \] (6)

Equation (6)/(5):

\[ \frac{(1 - \gamma)}{(1 - \gamma)} = \frac{B_a \left[ \gamma (\gamma - 1) \right] + \frac{2 \gamma}{\gamma - 1} (p - p_a)}{B_a \left( \gamma^2 - 1 \right) + p - p_a} = \frac{(1 + \gamma)}{(1 + \gamma)} = \frac{\gamma + 1}{\gamma} \]

\[ \gamma + 1 = \frac{\beta \left[ \gamma (\gamma - 1) \right] + 2 \gamma (\gamma - 1) (x / y - 1)}{\beta (\gamma + 1) + x - 1} \]

\[ x \gamma (z + 1) / (z - 1) - 1 + \beta (y - 1)^3 \]

\[ (x + 1) / (y - 1) - y \]
Dividing (5) by $R = p^2y^2$

$$\gamma n_{12}^2 = \beta (\gamma^2 - \gamma) + x^2 - \beta$$

Substituting $x$:

$$2(\gamma^2 - \gamma)y^2 + \gamma(x^2 - \gamma^2)y - x^2(x^2 - \gamma^2) + \beta = 0 \quad (3.53)$$

Given $M \rightarrow \text{only } y \rightarrow \text{and } x$