Lecture 9 - (Missing notes; derive Pacehek) Next class

Today: origin of magnetic fields

In ideal MHD, flux of magnetic field is constant: \( \oint \mathbf{B} \cdot d\mathbf{S} = \text{const} \)

So if ideal MHD is valid, I can know \( B \) (past), if I know it today.

However, we know that there is dissipation and that generation of \( B \) can happen:

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{v} \times \mathbf{B}) + \nu \nabla^2 \mathbf{B} - \frac{c}{n e^2} \mathbf{v} e \times \nabla p_e
\]

To calculate the diffusion time:

\[
\frac{\partial \mathbf{B}}{\partial t} - \nu_m \nabla^2 \mathbf{B} \Rightarrow \frac{\mathbf{B} - \nu_m \mathbf{B}}{60} \Rightarrow t_0 \sim \frac{R^2}{\nu_m}
\]

\[
\nu_m = \frac{\eta c^2}{4 \pi} \text{ (Spitzer)} \Rightarrow \nu_m \propto \frac{1}{T^{3/2}}
\]

<table>
<thead>
<tr>
<th>planet</th>
<th>galaxies</th>
</tr>
</thead>
<tbody>
<tr>
<td>( T(K) )</td>
<td>1000</td>
</tr>
<tr>
<td>( R^2 )</td>
<td>1</td>
</tr>
</tbody>
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So, for Earth: \( T = 300 \text{ K} \Rightarrow R = 1 \text{ m} \)
In the formation of stars, how does B get amplified?

\[ n = 10^4 \text{ cm}^{-3} \]
\[ B = n \times 10^{-5} \text{ G} \]

\[ B \propto r^{-2} \]

\[ B_{\text{final}} = B_i \frac{r_i^2}{r_f^2} = 2 \times 10^9 \text{ G} \text{ cannot be} \]

So what happens?

In the global \( r \) if you have conservation of angular momentum

\[ \frac{L^2}{\mu} = r^2 \times 7 \text{ if } L = \text{constant} \Rightarrow r = \text{constant} \]

\[ r^2 \cdot 2(r) = \text{constant} \]

\[ \Rightarrow 2(r) \propto r^{-2} \]

If \( r \) is very small \( B(r) \) is very large. If there is conservation of angular momentum, the centrifugal potential (the high spin of particles spiraling in the direction of a central object) constrains the particles to spiral in the direction of the central object. Conclusion: there is no conservation of total angular momentum.

- Keplerian relation:

\[ m \cdot v^2 = \mu \cdot \frac{1}{r} \Rightarrow v^2 = \frac{1}{\frac{1}{\mu} r} \]
As \( L \propto \left| \vec{r} \times \vec{p} \right| \propto r^2 B(r) \propto r^{1/2} \)

- There is transfer of angular momentum to the outside but still doesn't get to the observed values.

- A magnetic field parallel to rotation diminishes the angular momentum.

\[
\vec{F}_\phi = \frac{1}{c} \vec{J} \times \vec{B} = -\frac{\vec{J}_r B_z \phi}{c}
\]

Torque \( \vec{\tau} = \vec{r} \times \vec{F} = -r \vec{J}_r B_z \frac{\vec{L}}{c} = \frac{d\vec{L}}{dt} \)

- Diminish \( L \)

In a collapse: \( \rho \uparrow \rightarrow \) radio pe

- Exist scattering between neutral particles and electrons/ions
- Ambipolar diffusion

So, the magnetic flux is removed from the protostar

\( B \approx 1000 \, G \) (we have calculated \( 10^9 \, G \) for ideal MHD)
In the Sun: if we take into account the convection and diffusion, the field will be small.

What is not observed → so there is continuous regeneration of magnetic field!!!

- In convective stars (as the Sun): magnetic field are intense and variables → they are contemporaneous field regeneration

- In radiative stars: without convective motions → fossil magnetic field

→ Two classes of stars:
  A. favor an "original" field
  B. favor creation of magnetic field.

- In galaxies, too → tidal → original field
  When the B was formed?

Parker "B is not the original field in galaxies"

→ turbulent dissipation: anomalous viscosity (\(\nu_A\))
  \(L\) scale of turbulent vortices
General conclusion: not in stars, not in planets and not in galaxies—possess primordial magnetic field. There is advection, dissipation, regeneration.

6.1) Mechanisms of generation of fields

- Differential rotation
- Biermann battery
- Dynamo

Differential Rotation

$\nu = \nu(\rho) \rho$

Ideal MHD: ($\nu_m = 0$); in cylindrical coordinates

\[ \frac{\partial B_z}{\partial t} = 0 \]
\[ \frac{\partial B_y}{\partial t} - \frac{2ABr}{\partial t} \]

\[ A = R \frac{dV}{dr} \]

\[ B_p = B_p(t_0) + 2ABr(t - t_0) \]
Spherical Coordinates:

Polar field $B_p = B_r \hat{r} + B_\theta \hat{\theta}$

$B^2 = B_p + B^p$

If $B_p = cte$, then

$B_p \propto \frac{d}{dr} B_p$

So, a field line

Differential rotation: effectively don't create fields, only transform one component to the other. We will see it's important to amplify seed fields in dynamos.

Biermann battery
\[ p_e = nek_BT \rightarrow \nabla p_e = n\partial_e k_BT + nek_e \nabla T \]

\[ \nabla n_e \times \nabla p_e = (\nabla n_e \times \nabla T) nek_e \]

In a non-rotating body, \( \nabla n_e \parallel \nabla T \) and the term of the Biermann battery is zero.

What if it is rotating?

\[ \frac{\rho \nabla \vec{v}}{\partial t} - \rho \frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla) \vec{v} = -\nabla p + \rho \vec{g} + \vec{\Omega} \times \vec{B} \]

Hypothesis: azimuthal symmetry (cylindrical coordinates)

\[ \theta_t = 0 \]
\[ V_t = 0 \]
\[ V_\theta = c \theta e \]
\[ V_r = 2r \]

we are starting with NO FIELD

Remember: \( \frac{\partial \hat{\phi}}{\partial r} = -\hat{r} \)

Radial component: \( (\vec{v} \cdot \hat{r}) \hat{r} = v_r \hat{r} \cdot (V_r \hat{r} + V_\phi \hat{\phi}) = V_r \partial_r v_r + \frac{V_\phi}{r} \phi \)

Azimuthal component: \( (\vec{v} \cdot \hat{\phi}) v_\phi = \frac{v_\phi}{r} \frac{\partial \phi}{\partial r} = \frac{V_\phi}{r} V_r \phi \hat{r} - \frac{V_\phi}{r} \hat{r} \phi \)

Radial component of the equation of motion:

\[ \frac{\partial}{\partial t} (\rho \vec{v}) + \nabla \cdot (\rho \vec{v} \vec{v}) = -\nabla p + \rho \vec{g} + \vec{\Omega} \times \vec{B} \]
• Azimuthal component of the equation of motion:

\[
\frac{\partial \psi}{\partial t} + \frac{v_r \psi}{r} = - \frac{1}{\rho} \frac{\partial \rho}{\partial t} - \frac{1}{\rho} \frac{\partial \rho}{\partial r} \frac{\partial \psi}{\partial r} - \frac{\partial \psi}{\partial z} = 0
\]

\[
\frac{\partial \psi}{\partial t} = - \frac{1}{\rho} \frac{\partial \rho}{\partial t} \Rightarrow v_r \propto r^{-1} \Rightarrow \mathbf{F} \propto r^{-2}
\]

\[
\frac{\partial \psi}{\partial r}
\]

• Z-component of the equation of motion

\[
\frac{-1}{\rho} \frac{\partial \rho}{\partial t} - \frac{\partial \psi}{\partial z} = 0
\]

(6.2)

2 [6.1] and 2 [6.2]:

\[
\frac{\partial \rho}{\partial r} = \frac{\partial \rho}{\partial z}
\]

\[
-\frac{\partial}{\partial z} \frac{\partial^2 \rho}{\partial z^2} = -\frac{1}{\rho} \frac{\partial \rho}{\partial t} + \frac{1}{\rho} \frac{\partial \rho}{\partial z} \frac{\partial \rho}{\partial r} - \frac{\partial \psi}{\partial z} \frac{\partial \psi}{\partial r} - \frac{\partial \psi}{\partial z} \frac{\partial \psi}{\partial r} \frac{\partial \psi}{\partial r}
\]

\[
\frac{\partial}{\partial z} \left[ \frac{\partial \rho}{\partial z} \frac{\partial \rho}{\partial r} - \frac{\partial \rho}{\partial z} \frac{\partial \rho}{\partial r} \right] = \nabla \times \nabla \phi
\]

\[
\frac{\partial^2}{\partial z^2} = \frac{1}{\rho^2} \left[ \frac{\partial \rho}{\partial z} \frac{\partial \rho}{\partial r} - \frac{\partial \rho}{\partial z} \frac{\partial \rho}{\partial r} \right] = \frac{\nabla \times \nabla \phi}{\rho^2}
\]
\[ \rho = \frac{\vec{\nabla} n_e}{n_e} = \frac{\vec{\nabla} ne}{ne} \]

\[ -r \frac{\partial^2 \hat{\rho}}{\partial z^2} = \frac{k_e}{ne} (\vec{\nabla} n_e \times \vec{\nabla} T) \hat{\rho} \]

Because in the battery of Biermann, \( \frac{\partial \vec{B}}{\partial t} = -\frac{c}{ne^2e} (\vec{\nabla} n_e \times \vec{\nabla} pe) = -\frac{c \kappa_e}{ne} (\vec{\nabla} n_e \times \vec{\nabla} T) \)

\[
\frac{\partial \vec{B}}{\partial t} = \frac{c m r}{e} \frac{\partial^2 \hat{\rho}}{\partial r^2}
\]

Creation of magnetic field from nothing \( \Rightarrow \) we need only differential rotation.

\( \nabla \vec{B} \propto \hat{t} \Rightarrow B_y \propto 10^3 G \) for a star.

In the Sun: \( \frac{\partial \mu}{\partial z} \neq 0 \)

\[ \frac{\partial}{\partial z} \]

\[ \vec{w} \]

\[ \vec{w}_0 \]

Dynamo

- In convection stars, dynamo = motion in a flow that amplify
- In conductor fluid, a seed field
- Differential rotation
In stars we will apply this idea — we have already the necessary ingredients: material conductors, rotating system.

- Mechanism of Harrison (1970)

Origin of magnetic field in galaxies

- Photons of CMB interact with electrons and ions.
  - Thompson scattering $\propto m^{-2}$ — it's more efficient for electrons
  - Decoupling of electrons and ions

- Collapse of clouds creates drag forces — the clouds rotate.
  - The ions follow the rotational motion of the clouds, but electrons stay behind because they are more coupled to the photons

$$\mathbf{j}_f = ne \mathbf{e} (V_i - V_e)$$  \{ create & field \}

$$\mathbf{4} \mathbf{j}_f = \mathbf{\nabla} \times \mathbf{B}_P$$

$$\frac{\partial \mathbf{B}_P}{\partial t} = -\mathbf{c} \mathbf{\nabla} \times \mathbf{E}$$

$$\rightarrow \frac{1}{c} \mathbf{\partial} \left( \int \mathbf{B}_P^2 \, d\mathbf{a} \right) = \oint \mathbf{E} \cdot d\mathbf{l}$$

Flux
Field created by the friction of the electrons (generate current and then generate $B$)

$$\frac{\text{d}N_e}{\text{d}t} = -\frac{\text{d}i}{\text{d}t} + N_e \frac{\text{d}e}{\text{d}t} - N_e e \left( E + \frac{\text{v}_e x B}{c} \right) + \text{Pe} + \text{Pe}$$

$B_p \propto$ affirming that $\text{d}v_e/\text{d}t = 0$  

$NeE = \text{Pe}$

$Pe = m_e n_e (\nu_{re}) v_T \sim Ne e E_x$

$\gamma_T$: rate of scattering

$\nu_{re} \sim w_r$

$\gamma_T \propto \frac{\text{d} \nu_{re} e \text{c} t}{\nu}$

$e_0$ density of energy of the radiation field (photons)

$$E_0 = \frac{m_e \nu_{re} w_r e \text{c}^2}{e} \frac{\nu_{\text{c}}}{\nu e} \left( \frac{4}{3} \right)$$

$$B_p \sim \frac{8}{3} \frac{\text{P}_T}{e} \left( \int \nu u \text{d}t \right)$$

$\sim$ Big Bang

Hypothesis: $\lambda = 10$ the galaxies had $\nu_p$ equal of what is observed today and the age of the universe was $10^{16}$ sec, so $\nu = 4 \times 10^9 \text{erg} / \text{cm}^3$
The observed radiation of the Ly α forest is polarized (Faraday rotation), so we can estimate $B$.

$B_{\text{observed}} \approx 10^{15} \Rightarrow e^{35} \approx e^{nt}$ \hspace{1cm} \text{n: rate of growth of B field.}

In the galaxy - is there a rate of growth of $B$ today? Is $n_{\text{today}} \approx 0$? It's believed that the galaxy is in MHD equilibrium. This gives a value of $B \approx 10^{-6}$ as it is observed.

If $B > 10^{-6}$, $\Rightarrow \Delta z \left( \frac{p+B^2}{8\pi} \right) > 1/4$

If $B$ will grow until today: $n^{-1} \approx 3 \times 10^8$ years.

Because we believe that $n_{\text{today}} \approx 0$, the growth of $B$ stopped in the past $\Rightarrow n^{-1} < 3 \times 10^8$ years.

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Mechanism of Vilinkin & Vachaspati.
→ mechanism of Harrison
→ seed field even before \( z = 10 \)

- Mechanism of Turner & Widrow (1988)

In the inflation period \( \rightarrow \) vacuum (change instantaneously) with electric field \( \rightarrow \) create \( B \)

6.2 Types of Dynamos

(I) Homopolar Dynamo
(II) Dynamo of Heisenberg
(III) Dynamo of Zeldovich
(IV) Dynamo of Parker

(I) \( \oint \vec{B} \cdot dl = \frac{I y \vec{n}}{c} \)

\[ E = -\frac{\vec{v} \times \vec{B}}{c} = -\frac{\omega r}{c} \frac{\vec{B}(r)}{r} \]

\( V = \oint \frac{\vec{E} \cdot dr}{c} = -\omega \left( r \vec{B} \cdot dr \right) \]

(\( V_0 = \omega r \))

(\( V \) self-induction)