Lecture 2 - Waves and Instabilities

In flows, we have instabilities such as Rayleigh-Taylor, Kelvin-Helmholtz, and thermal instabilities. These same instabilities persist in a modified form in plasmas. Beyond that, in plasmas, Alfvén waves appear as well.

Both instabilities as well as waves can be studied in the linear regime where the perturbations are assumed small. Waves correspond to real frequencies, while instabilities correspond to complex frequencies with a negative imaginary part.

2.1 Dispersion Relationship

Basic equations are:

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{u} \times \mathbf{B}) + \frac{\mathbf{J} \times \mathbf{B}}{\mu_0} \tag{2.1}
\]

• Induction equation (with \( \mathbf{\nu} = \mathbf{B} \))

\[
\frac{\partial \mathbf{B}}{\partial t} + (\mathbf{\nu} \cdot \nabla) \mathbf{B} + \nabla \times (\mathbf{\epsilon} \times \mathbf{B}) = 0 \tag{2.2}
\]

• Ohm's law:

\[
\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{\nu} \times \mathbf{B}) + \mathbf{\epsilon} \times \mathbf{B} = \frac{4}{3} \nabla \cdot \mathbf{J} \tag{2.3}
\]

\[
= \frac{1}{\mathbf{\mu}_0} \left( \frac{\partial \mathbf{J}}{\partial t} + \nabla \times (\mathbf{\nu} \times \mathbf{J}) \right) = 0
\]
Let's take $S = \frac{v}{\rho}$ and $\rho = \rho_0 T / m$

We can evolve $\rho$, $\rho$, and $T$ if $T$ is known.

For that we can use the equation of energy conservation or the heat conduction equation:

$$ S = \frac{k}{m} \ln \left( \frac{1}{1 - \frac{\rho}{\rho_0}} \right) = \frac{k_0}{m} \left( \frac{1}{\rho_0} \ln T - \ln \rho \right) $$

where $S = dP / dV$

for ideal gas $v = \frac{S}{k}$

$$ \frac{k_0}{m} \left( \frac{1}{\rho_0} \frac{\partial \rho}{\partial T} + \frac{1}{\rho_0} \frac{\partial^2 T}{\partial \rho \partial T} \right) $n

$$ -K \frac{\partial^2 T}{\partial \rho^2} - \frac{4}{5} \left( \frac{\partial T}{\partial \rho} \right)^2 + \rho \frac{\partial (\rho T)}{\partial T} = 0 $$

(0 rests to page 2)

Equation of energy

[For more, see page]
So we will consider a state of zero-order \( (\mathbf{B}_0, \mathbf{P}_0, \mathbf{v}_0, \mathbf{T}_0) \) and study perturbations of the type

\[
(\mathbf{B}, \mathbf{P}) = \mathbf{B}_0 + \xi_1(\mathbf{x}, \mathbf{P}) + \xi_2(\mathbf{y}, \mathbf{P}) + \cdots \tag{2.6}
\]

where \( \xi \) is a perturbation of first order, and we will ignore all perturbations of higher order such as \( \mathbf{B}^2, \mathbf{P}^2, \mathbf{y}_x, \mathbf{y}_y, \mathbf{y}_z, \mathbf{z}_x, \mathbf{z}_y, \mathbf{z}_z, \mathbf{z}_x, \mathbf{z}_y, \mathbf{z}_z \), etc.

The equilibrium state is \( (\mathbf{B}_0, \mathbf{P}_0, \mathbf{v}_0, \mathbf{T}_0) \). We will assume \( \mathbf{B} = 0 \) and \( \mathbf{P}_0 = 0 \).

Substituting (2.6) in (2.1), (2.2), (2.3) and (2.5) and keeping only the terms that do not go to zero or aren't negligible, we have that

\[
\frac{\partial \mathbf{B}_0}{\partial t} + \mathbf{B}_0 \cdot \nabla \mathbf{B}_0 + \nabla \times \mathbf{B}_0 = \frac{\mathbf{B}_0}{\mathbf{v}_0} + \mathbf{B}_0 \nabla \mathbf{v}_0 + \mathbf{B}_0 \mathbf{v}_0 + \mathbf{B}_0 \nabla \mathbf{v}_0 + \frac{\partial \mathbf{B}_0}{\partial t} = 0
\]

\[
\frac{\partial \mathbf{B}_0}{\partial t} + \mathbf{B}_0 \cdot \nabla \mathbf{B}_0 + \nabla \times \mathbf{B}_0 = 0 \tag{2.7}
\]

\[
\frac{\partial \mathbf{v}_0}{\partial t} + \mathbf{v}_0 \cdot \nabla \mathbf{v}_0 + \frac{\mathbf{B}_0}{\mathbf{v}_0} = 0 \tag{2.8}
\]

\[
\frac{\partial \mathbf{v}_0}{\partial t} + \mathbf{v}_0 \cdot \nabla \mathbf{v}_0 = 0 \tag{2.9}
\]

\[
\frac{\partial \mathbf{v}_0}{\partial t} + \mathbf{v}_0 \cdot \nabla \mathbf{v}_0 = 0 \tag{2.10}
\]
\[ 2 \frac{\partial \rho}{\partial t} + \frac{1}{\rho_0} \frac{\partial}{\partial r} \left( \rho \frac{\partial r}{\partial t} \right) - \frac{1}{\rho_0} \frac{\partial}{\partial t} \left( \rho \frac{\partial p}{\partial r} \right) = 0 \]  \tag{2.9}

\[ \frac{1}{m} \left( \frac{1}{\rho_0} \frac{\partial}{\partial t} + \frac{1}{\rho_0} \frac{\partial}{\partial r} \right) \left( \rho \frac{\partial p}{\partial r} \right) - \nabla \cdot \left( \rho \frac{\partial \mathbf{u}}{\partial t} \right) = 0 \]  

(Like term \((\mathbf{v} \cdot \nabla)\mathbf{v}\))

\[ \frac{1}{m} \left( \frac{1}{\rho_0} \frac{\partial}{\partial t} - \frac{1}{\rho_0} \frac{\partial}{\partial r} \right) \left( \rho \frac{\partial p}{\partial r} \right) - \nabla \cdot \left( \rho \frac{\partial \mathbf{u}}{\partial t} \right) = 0 \]  \tag{2.10}

where

\[ f = \frac{\partial u}{\partial t} \]  \tag{2.11}

\[ f_T = \frac{\partial u_T}{\partial t} \]  \tag{2.12}

\( u, v, \mathbf{v}, \mathbf{u}, \mathbf{u}_T \) are in zero-order and are constant being in order zero.

Note that all the terms \((\mathbf{u} \cdot \nabla)\mathbf{u}\) as well as viscous dissipation \((\nabla \cdot \mathbf{u} T)\) do not make because they have second order in \( \mathbf{u} \) and \( \mathbf{u}_T \).
The fact that the eqs. (2.12) - (2.13) poses constant coefficients, 
their solution are expressible as:

$$f_i(x,t) = \exp(at + ikx)$$  \hspace{1cm} (2.13)

When we use a solution like that we are focusing in one single 
component of wave of $k = \omega/k$ valid in any time $t$, and we are 
looking implicitly for positive values of the $\omega$ (growth) to represent 
an instability. However, $n^2$ can be imaginary, representing a wave 
of amplitude or complex, representing a damped wave or a growing 
wave.

We want to obtain the "dispersion relation", $n(k)$

Considering \text{(2.13)}, we can write:

$$\frac{\partial f_i(x,t)}{\partial t} = a f_i(x,t) \exp(at + ikx) = n^2 f_i(x,t)$$  \hspace{1cm} (2.14)

$$\frac{\partial^2 f_i(x,t)}{\partial x^2} = k^2 f_i(x,t) \exp(at + ikx) = k^2 f_i(x,t)$$  \hspace{1cm} (2.15)

$$\frac{\partial f_i(x,t)}{\partial x} = ik f_i(x,t)$$  \hspace{1cm} (2.16)

$$\frac{\partial^2 f_i(x,t)}{\partial x^2} = ik^2 f_i(x,t)$$  \hspace{1cm} (2.17)

To have an instability, $n_R \neq 0$

have a wave $n_i = n - i$, constant amplitude 
meaning amplitude not constant.
In the equation (2.10),

\[ \left( \frac{k_a}{m} \frac{\bar{p}_T}{T_0} + \frac{\rho_0}{T_0} \frac{\bar{p}_0}{\rho_0} \frac{\bar{T}_0}{T_0} \right) - \frac{k_a}{m} T_0 = 0 \]

\[ \left( \frac{k_a}{m} \frac{\bar{p}_T}{T_0} + \frac{\rho_0}{T_0} \frac{\bar{p}_0}{\rho_0} \frac{\bar{T}_0}{T_0} \right) \frac{\bar{T}_0}{T_0} - \frac{k_a}{m} T_0 \frac{\bar{p}_T}{T_0} + \bar{p}_0 \frac{\rho_0}{T_0} \frac{\bar{T}_0}{T_0} = 0 \]

\[ \frac{\bar{p}_T}{T_0} = \left[ \frac{n k_a \bar{p}_T - \rho_0 \rho_0 \frac{\bar{p}_0}{\rho_0} \frac{\bar{T}_0}{T_0}}{\frac{k_a}{m} \frac{\bar{p}_T}{T_0} + \frac{\rho_0}{T_0} \frac{\bar{p}_0}{\rho_0} \frac{\bar{T}_0}{T_0}} \right] \]

\[ \frac{T_0}{\bar{T}_0} = \frac{\rho_0}{\rho} \frac{\frac{k_a}{m} \frac{\bar{p}_T}{T_0} - \frac{\rho_0}{T_0} \frac{\bar{p}_0}{\rho_0} \frac{\bar{T}_0}{T_0}}{\frac{k_a}{m} \frac{\bar{p}_T}{T_0} + \frac{\rho_0}{T_0} \frac{\bar{p}_0}{\rho_0} \frac{\bar{T}_0}{T_0}} \]

K: Thermal conductivity coefficient
k: Wavenumber that appear in (2.13)

\[ k_a = \text{Bullermann constant} \]

Since \( \rho = \rho_0 \frac{T_0}{T} = \frac{T_0 + T_0}{T} \)

\[ \frac{\bar{p}_T}{\bar{T}_0} = \frac{k_a}{m} \left( \rho \frac{\bar{p}_T}{T_0} + \frac{\bar{p}_0}{\rho_0} \frac{\bar{T}_0}{T_0} \right) = \Delta p + \Delta \rho \]

So \( \frac{\Delta p}{\bar{p}} = \frac{\Delta \rho}{\rho} = \frac{\Delta T}{T} \quad \Rightarrow \quad \bar{p}_T = \bar{p}_0 - \frac{T_0}{\rho_0} \frac{\bar{p}_0}{\rho_0} \frac{\bar{T}_0}{T_0} \)
So \[ \rho = \rho_0 + \rho_v \left( \frac{\sigma(n,k)}{\rho_0} \right) \]

If \( I_p = I_v = 0 \Rightarrow \sigma(n,k) = 1 \) for any \( n,k \)

\[ \sigma = \frac{\sigma}{\rho_0} \quad \text{or} \quad \rho = \rho_0 \sigma \]

Therefore, for \( I_p = I_v = 0 \) the perturbation will not involve any variation of density.

Or \[ \rho = \rho_0 \sigma(n,k) \] (2.19)

Doing the same thing for Eq. (2.7) - (2.9) we get

\[ n^2 \sigma - 1 \left( B_0 \right)^2 + \beta \sigma \left( B_0 \right)^2 + \frac{\kappa}{\rho_0} \sigma \left( B_0 \right)^2 = 0 \]

\[ \beta \sigma(n^2 + k^2) = i \left( B_0 \right)^2 - i \left( B_0 \right)^2 \] (2.20)

And \[ \frac{n_0^2 + i \kappa \nu}{\rho_0} = 0 \Rightarrow \rho_0 = -i \frac{\kappa \nu}{n} \] (2.21)

Also note that \[ \frac{i}{m} \frac{d}{dt} \tilde{V} = \frac{k_0 \kappa}{m} \tilde{V} = \frac{\kappa}{m} \tilde{V} \]

\[ \tilde{V} = n_0 \tilde{V} - \frac{\rho_0}{\rho} \eta - \frac{\rho_0 \rho_v}{\rho} \eta \]

\[ \frac{\rho_0}{\rho} = \frac{\rho_0}{\rho} \]
\[ v^2 + \frac{i}{\mu_0} k_0^2 \frac{\partial^2 v}{\partial t^2} + i \frac{\partial}{\partial x} \times \hat{B}_0 \times (\hat{B}_0 \times \hat{B}_0) + i \gamma \frac{\partial}{\partial t} \hat{D}(\hat{B}_0 \times \hat{B}_0) = 0 \]

(2.22)

If we eliminate \( v_a \) and \( B_1 \) using (2.20) and (2.21), we obtain an equation just for \( \hat{v} \):

\[
\begin{aligned}
v^2 + \gamma \frac{\partial}{\partial t} \hat{D}(\hat{B}_0 \times \hat{B}_0) + \frac{1}{\mu_0} \frac{\partial^2 \hat{v}}{\partial t^2} + \frac{\partial}{\partial x} \times \left[ \hat{B}_0 \times \left( \hat{B}_0 \times \hat{B}_0 \right) \right] &= 0 \\
&\text{where } \gamma = \left( \frac{\mu_0}{\mu_0} \right)^{1/2}
\end{aligned}
\]

(2.23)

This equation is actually three equations, for the three components of \( \hat{v} \), where each component is coupled to \( \hat{B}_0 \).

Let choose a coordinate system:

\[
\hat{v} = (v_x, v_y, v_z)
\]

\[
\hat{B}_0 \parallel (1)
\]

\[
\hat{B}_0 \times \hat{B}_0 \parallel (2)
\]

\[
\hat{B}_0 \cdot \hat{B}_0 = k \theta_0 \cos \beta
\]

\[
\hat{B}_0 \cdot \hat{v} = k v_x
\]

\[
\hat{B}_0 \times \hat{B}_0 = \gamma \left( v_x \hat{e}_y - v_y \hat{e}_x \right)
\]

\[
\hat{B}_0 \cdot \hat{B}_0 = k \hat{B}_0 \cos \beta
\]

\[
\hat{B}_0 \times \hat{B}_0 = -k \hat{B}_0 \sin \beta
\]

\[
\hat{B}_0 \cdot \hat{B}_0 = k \hat{B}_0 \cos \beta
\]

\[
\hat{B}_0 \times \hat{B}_0 = -k \hat{B}_0 \sin \beta
\]
22. Alien waves

\[ \text{From (2.31), } v_2 \text{ is not coupled to the other two velocity components.} \]

\[ v_2 \left[ n + \frac{k v_2 \cos \theta}{n + k v_2} \right] = 0 \]

for any \( n, k \).

\[ n (n + k v_2) + k v_2 \cos \theta + \sqrt{n (n + k v_2)} = 0 \]

\[ n^2 + n (n + k v_2) + k^2 v_2 \cos \theta + k v_2 = 0 \]

\[ n = -\frac{1}{2} \left( (n + k v_2) \pm i k v_2 \cos \theta \right) \left[ 1 - \frac{k^2 (n + k v_2)}{4 n v_2 \cos \theta} \right]^{1/2} \]  \[(2.31)\]

In the limit that \( k v_2 \ll n \) and \( k v_2 \ll v_2 \),

\[ n = \pm i \frac{k v_2 \cos \theta}{2} \]

Because \( n \) is purely imaginary, I have a wave.

Please quote: \( V_2 = \frac{k v_2 \cos \theta}{2} \).

\[ \text{Leqne 3. [Important note: October 04: Midterm]} \]

\[ \text{[HW/#1 due next week]} \]

Remember our condition on \( x = \frac{k^2 d}{v_2} \) to \( v_2 \), so Alien waves are a transverse wave to the motion \( \rightarrow \) it doesn't cause compression in the fluid. Why?

Compression: \( \nabla \cdot \mathbf{v} = \frac{1}{k_0} \cdot \mathbf{v} = 0 \)

\[ v_2 \]

\[ \text{analog for a} \text{ in a fluid} \]

\[ \frac{v_2}{\lambda} \text{ the magnetic tension} \]

\[ \frac{B^2}{2\mu} \text{ the reduced wave} \]