Exam 3 Topics

• Faraday’s Law
• Self Inductance
  • Energy Stored in Inductor/Magnetic Field
• Circuits
  • LR Circuits
  • Undriven (R)LC Circuits
  • Driven RLC Circuits
• Displacement Current
• Poynting Vector

NO: B Materials, Transformers, Mutual Inductance, EM Waves
General Exam Suggestions

• You should be able to complete every problem
  • If you are confused, ask
  • If it seems too hard, you aren’t thinking enough
  • Look for hints in other problems
  • If you are doing math, you’re doing too much
• Read directions completely (before & after)
• Write down what you know before starting
• Draw pictures, define (label) variables
  • Make sure that unknowns drop out of solution
• Don’t forget units!
Maxwell's Equations

\[ \oint_{S} \mathbf{E} \cdot d\mathbf{A} = \frac{Q_{\text{in}}}{\varepsilon_{0}} \]  
(Gauss's Law)

\[ \oint_{C} \mathbf{E} \cdot d\mathbf{s} = -\frac{d\Phi_{B}}{dt} \]  
(Faraday's Law)

\[ \oiint_{S} \mathbf{B} \cdot d\mathbf{A} = 0 \]  
(Magnetic Gauss's Law)

\[ \oint_{C} \mathbf{B} \cdot d\mathbf{s} = \mu_{0}I_{\text{enc}} + \mu_{0}\varepsilon_{0} \frac{d\Phi_{E}}{dt} \]  
(Ampere-Maxwell Law)

\[ \mathbf{F} = q(\mathbf{E} + \mathbf{v} \times \mathbf{B}) \]  
(Lorentz force Law)
Gauss’s Law:

\[ \iiint \vec{E} \cdot d\vec{A} = \frac{q_{in}}{\varepsilon_0} \]

**Spherical Symmetry**

- Spherical Gaussian Pillbox

**Cylindrical Symmetry**

- Cylindrical Gaussian Pillbox

**Planar Symmetry**

- Planar Gaussian Pillbox
Ampere’s Law: \[ \int \vec{B} \cdot d\vec{s} = \mu_0 I_{enc} \]

- Long Circular Symmetry
- (Infinite) Current Sheet
- Torus/Coax
- Solenoid = 2 Current Sheets
Faraday’s Law of Induction

\[ \mathcal{E} = \int \vec{E} \cdot d\vec{s} = -N \frac{d\Phi_B}{dt} \]

Induced EMF is in direction that opposes the change in flux that caused it

Lenz’s Law:

Moving bar, entering field

Ramp B

Rotate area in field

Induced EMF is in direction that **opposes** the change in flux that caused it
Self Inductance & Inductors

L = \frac{N\Phi}{I}

When traveling in direction of current:

\[ \mathcal{E} = -L \frac{dI}{dt} \]

Notice: This is called “Back EMF”
It is just Faraday’s Law!
Energy Stored in Inductor

\[ U_L = \frac{1}{2} L I^2 \]

Energy is stored in the magnetic field:

\[ u_B = \frac{B^2}{2\mu_0} \quad : \text{Magnetic Energy Density} \]
LR Circuit

\[ \mathcal{E} - IR - L \frac{dI}{dt} = 0 \]

Readings on Voltmeter
- Inductor (a to b)
- Resistor (c to a)

\[ \tau = \frac{L}{R} \]

\[ 0.36 \frac{\mathcal{E}}{R} \]

- \( t = 0^+ \): Current is trying to change. Inductor works as hard as it needs to in order to stop it.
- \( t = \infty \): Current is steady. Inductor does nothing.
General Comment: LR/RC

All Quantities Either:

\[ \text{Value}(t) = \text{Value}_{\text{Final}} \left(1 - e^{-t/\tau}\right) \]

\[ \text{Value}(t) = \text{Value}_0 e^{-t/\tau} \]

\( \tau \) can be obtained from differential equation (prefactor on \( d/dt \)) e.g. \( \tau = L/R \) or \( \tau = RC \)
Undriven LC Circuit

Oscillations: From charge on capacitor (Spring) to current in inductor (Mass)

$$\omega_0 = \frac{1}{\sqrt{LC}}$$
Damped LC Oscillations

Resistor dissipates energy and system rings down over time

\[ Q = \pi n = \frac{\omega L}{R} \]
## AC Circuits: Summary

<table>
<thead>
<tr>
<th>Element</th>
<th>V vs $I_0$</th>
<th>Current vs. Voltage</th>
<th>Resistance-Reactance (Impedance)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resistor</td>
<td>$V_{0R} = I_0R$</td>
<td>In Phase</td>
<td>$R = R$</td>
</tr>
<tr>
<td>Capacitor</td>
<td>$V_{0C} = \frac{I_0}{\omega C}$</td>
<td>Leads (90°)</td>
<td>$X_C = \frac{1}{\omega C}$</td>
</tr>
<tr>
<td>Inductor</td>
<td>$V_{0L} = I_0\omega L$</td>
<td>Lags (90°)</td>
<td>$X_L = \omega L$</td>
</tr>
</tbody>
</table>
Now Solve: \[ V_S = V_R + V_L + V_C \]

Now we just need to read the phasor diagram!
Driven RLC Series Circuit

\[ I(t) = I_0 \sin(\omega t - \varphi) \]

\[ V_S = V_{0S} \sin(\omega t) \]

\[ V_{S0} = \sqrt{V_{R0}^2 + (V_{L0} - V_{C0})^2} = I_0 \sqrt{R^2 + (X_L - X_C)^2} \equiv I_0 Z \]

\[ I_0 = \frac{V_{S0}}{Z} \]

\[ Z = \sqrt{R^2 + (X_L - X_C)^2} \]

\[ \phi = \tan^{-1}\left( \frac{X_L - X_C}{R} \right) \]

Impedance
Plot I, V’s vs. Time

\[ I(t) = I_0 \sin (\omega t) \]

\[ V_R(t) = I_0 R \sin (\omega t) \]

\[ V_L(t) = I_0 X_L \sin (\omega t + \frac{\pi}{2}) \]

\[ V_C(t) = I_0 X_C \sin (\omega t - \frac{\pi}{2}) \]

\[ V_S(t) = V_{S0} \sin (\omega t + \phi) \]

\[ \phi = \tan^{-1} \left( \frac{X_L - X_C}{R} \right) \]
Resonance

\[ I_0 = \frac{V_0}{Z} = \frac{V_0}{\sqrt{R^2 + (X_L - X_C)^2}} \]; \quad X_L = \omega L, \quad X_C = \frac{1}{\omega C} \]

On resonance:
\( I_0 \) is max; \( X_L = X_C; \ Z = R; \)
\( \phi = 0; \ \text{Power to} \ R \ \text{is max} \)

C-like:
\( \phi < 0 \)
\( I \) leads

L-like:
\( \phi > 0 \)
\( I \) lags
Average Power: Resistor

\[ < P > = < I^2(t)R > \]

\[ = < I_0^2 \sin^2(\omega t - \phi)R > \]

\[ = I_0^2 R < \sin^2(\omega t - \phi) > \]

\[ = I_0^2 R \left( \frac{1}{2} \right) \]
Problem 1: RLC Circuit

Consider a circuit consisting of an AC voltage source: \( V(t)=V_0\sin(\omega t) \) connected in series to a capacitor \( C \) and a coil, which has resistance \( R \) and inductance \( L_0 \).

1. Write a differential equation for the current in this circuit.
2. What angular frequency \( \omega_{res} \) would produce a maximum current?
3. What is the voltage across the capacitor when the circuit is driven at this frequency?
Solution 1: RLC Circuit

1. Differential Eqn:
\[ V_S - IR - L \frac{dI}{dt} - \frac{Q}{C} = 0 \]
\[ \frac{dI}{dt} R + L \frac{d^2 I}{dt^2} + \frac{I}{C} = \frac{d}{dt} \frac{V_S}{\omega} \]
\[ = \omega V_0 \cos(\omega t) \]

2. Maximum current on resonance:
\[ \omega_{res} = \frac{1}{\sqrt{L_0 C}} \]
3. Voltage on Capacitor

\[ V_{C0} = I_0 X_C \]

What is \( I_0, X_C \)?

\[ I_0 = \frac{V_0}{Z} = \frac{V_0}{R} \text{ (resonance)} \]

\[ X_C = \frac{1}{\omega C} = \frac{\sqrt{L_0 C}}{C} = \sqrt{\frac{L_0}{C}} \]

\[ V = V_{C0} \left( - \cos (\omega t) \right) = V_{C0} \sin (\omega t - \frac{\pi}{2}) \]
Problem 1, Part 2: RLC Circuit

Continue considering that LRC circuit. Insert an iron bar into the coil. Its inductance changes by a factor of 5 to $L = L_{core}$

4. Did the inductance increase or decrease?
5. Is the new resonance frequency larger, smaller or the same as before?
6. Now drive the new circuit with the original $\omega_{res}$. Does the current peak before, after, or at the same time as the supply voltage?
Solution 1, Part 2: RLC Circuit

4. Putting in an iron core INCREASES the inductance

5. The new resonance frequency is smaller

6. If we drive at the original resonance frequency then we are now driving ABOVE the resonance frequency. That means we are inductor like, which means that the current lags the voltage.
Problem 2: Self-Inductance

The above inductor consists of two solenoids (radius $b$, $n$ turns/meter, and radius $a$, $3n$ turns/meter) attached together such that the current pictured goes counter-clockwise in both of them according to the observer.

What is the self inductance of the above inductor?
Solution 2: Self-Inductance

Inside Inner Solenoid:
\[ \int \vec{B} \cdot d\vec{s} = Bl = \mu_0 \left( nlI + 3nlI \right) \]
\[ \Rightarrow B = 4\mu_0 nI \]

Between Solenoids:
\[ B = \mu_0 nI \]

\[ U = \frac{B^2}{2\mu_0} \cdot \text{Volume} \]
\[ = \frac{(4\mu_0 nI)^2}{2\mu_0} \pi a^2 \ell + \frac{(\mu_0 nI)^2}{2\mu_0} \pi \left( b^2 - a^2 \right) \ell \]
Solution 2: Self-Inductance

\[ U = \frac{(\mu_0 n I)^2}{2\mu_0} \pi \ell \left\{ 15a^2 + b^2 \right\} \]

\[ U = \frac{1}{2} L I^2 \]

\[ \Rightarrow L = \frac{(\mu_0 n)^2}{\mu_0} \pi \ell \left\{ 15a^2 + b^2 \right\} \]

Could also have used: \[ L = \frac{N\Phi}{I} \]
Problem 3: Pie Wedge

Consider the following pie shaped circuit. The arm is free to pivot about the center, P, and has mass \( m \) and resistance \( R \).

1. If the angle \( \theta \) decreases in time (the bar is falling), what is the direction of current?
2. If \( \theta = \theta(t) \), what is the rate of change of magnetic flux through the pie-shaped circuit?
Solution 3: Pie Wedge

1) Direction of I?

Lenz’s Law says: try to oppose decreasing flux

I Counter-Clockwise (B out)

2) $\theta = \theta(t)$, rate of change of magnetic flux?

$$A = \pi a^2 \left( \frac{\theta}{2\pi} \right) = \frac{\theta a^2}{2}$$

$$\frac{d\Phi_B}{dt} = \frac{d}{dt}(BA) = B \frac{d}{dt} \frac{\theta a^2}{2}$$

$$= \frac{Ba^2}{2} \frac{d\theta}{dt}$$
Problem 3, Part 2: Pie Wedge

3. What is the magnetic force on the bar (magnitude and direction – indicated on figure)

4. What torque does this create about P? (HINT: Assume force acts at bar center)
Solution 3, Part 2: Pie Wedge

3) Magnetic Force?
\[ d\vec{F} = I d\vec{s} \times \vec{B} \]
\[ F = I a B \]
\[ I = \frac{\varepsilon}{R} = \frac{1}{R} \frac{d\Phi_B}{dt} = \frac{1}{R} \frac{B a^2}{2} \frac{d\theta}{dt} \]
\[ F = \frac{B^2 a^3}{2R} \frac{d\theta}{dt} \]
(Dir. as pictured)

4) Torque?
\[ \vec{\tau} = \vec{r} \times \vec{F} \Rightarrow \tau = \frac{a}{2} F = \frac{B^2 a^4}{4R} \frac{d\theta}{dt} \]
(out of page)
Problem 4: RLC Circuit

1. What energy is currently stored in the magnetic field of the inductor?

2. At time $t = 0$, the switch $S$ is thrown to position $b$. By applying Faraday's Law to the bottom loop of the above circuit, obtain a differential equation for the behavior of charge $Q$ on the capacitor with time.

The switch has been in position $a$ for a long time. The capacitor is uncharged.
Solution 4: RLC Circuit

1. Energy Stored in Inductor

\[ U = \frac{1}{2} L I^2 = \frac{1}{2} L \left( \frac{\varepsilon}{R} \right)^2 \]

2. Write Differential Equation

\[ -L \frac{dI}{dt} - \frac{Q}{C} = 0 \]

\[ I = \frac{dQ}{dt} \Rightarrow L \frac{d^2Q}{dt^2} + \frac{Q}{C} = 0 \]
3. Write down an explicit solution for $Q(t)$ that satisfies your differential equation above and the initial conditions of this problem.

4. How long after $t = 0$ does it take for the electrical energy stored in the capacitor to reach its first maximum, in terms of the quantities given? At that time, what is the energy stored in the inductor? In the capacitor?
Solution 4: RLC Circuit

3. Solution for $Q(t)$: $Q(t) = Q_{\text{max}} \sin(\omega t)$

\[
\omega = \frac{1}{\sqrt{LC}} \quad \text{and} \quad \omega Q_{\text{max}} = I_0 = \frac{\varepsilon}{R} \quad \Rightarrow \quad Q_{\text{max}} = \frac{\varepsilon\sqrt{LC}}{R}
\]

4. Time to charge capacitor

\[
T = \frac{2\pi}{\omega} = 2\pi\sqrt{LC} \quad \Rightarrow \quad T_{\text{Charge}} = \frac{T}{4} = \frac{\pi\sqrt{LC}}{2}
\]

Energy in inductor = 0

Energy in capacitor = Initial Energy:

\[
U = \frac{1}{2} L \left( \frac{\varepsilon}{R} \right)^2
\]