The eigenfunction-eigenvalue problem can be written as

$$\hat{D} f(x) = \lambda f(x)$$  \hspace{1cm} \text{(i)}$$

where $\lambda$ is the eigenvalue.

Using the defn of $\hat{D}$, (i) can be written as

$$f(x+\delta) = \lambda f(x)$$  \hspace{1cm} \text{(ii)}$$

In order to show that the eigenfunctions of $\hat{D}$ are of the form $\varphi(x) = e^{\delta x} g(x)$, we need (where $g(x)$ satisfies $g(x+\delta) = g(x)$), we need to check if $\varphi(x)$ satisfies (ii).

Let $f(x) = e^{\delta x} g(x)$

$$f(x+\delta) = e^{\delta (x+\delta)} g(x+\delta)$$

$$= e^{\delta \delta} e^{\delta x} g(x+\delta)$$

$$= e^{\delta \delta} (e^{\delta x} g(x)) \left[\text{using } g(x+\delta) = g(x)\right]$$

$$= e^{\delta \delta} f(x)$$  \hspace{1cm} \text{(iii)}$$

Comparing (ii) & (iii) we see that the eigenvalue $\lambda = e^{\delta \delta}$, and $f(x) = e^{\delta x} g(x)$ is an eigenfunction corresponding to that eigenvalue.
Our equation is
\[
\frac{\partial^2 \Psi}{\partial t^2} = \hat{H} \Psi
\]  
--- (i)

Now, \( \hat{H}(m, n) = \hat{H}_1(m) + \hat{H}_2(n) \)  
--- (ii)

and \( \Psi(m, n, t) = \psi_1(m, t) \psi_2(n, t) \)  
--- (iii)

We need to show, (iii) is a solution of (i), when \( \hat{H} \) is given by (ii)

Left side of (i) ---
\[
\frac{\partial^2 \Psi}{\partial t^2} = \frac{\partial}{\partial t} \left( \frac{\partial \psi}{\partial t} \right) \left( \psi_1(m, t) \psi_2(n, t) \right)
\]
\[
= \psi_1 \frac{\partial^2 \psi}{\partial t^2} + \psi_2 \frac{\partial^2 \psi}{\partial t^2}
\]
\[
= \frac{\partial}{\partial t} \left( \psi_1 \frac{\partial \psi}{\partial t} \right) + \frac{\partial}{\partial t} \left( \psi_2 \frac{\partial \psi}{\partial t} \right)
\]
\[
= \frac{\partial}{\partial t} \left( \psi_1 \frac{\partial \psi}{\partial t} + \psi_2 \frac{\partial \psi}{\partial t} \right)
\]

--- (iv)

Now, \( \psi_2 \) is a function of \( n_2 \) and \( t \) only

and, \( \psi_1 \) is a function of \( m_1 \) and \( t \).

hence, \( \hat{H}_2 \psi_2 = \hat{H}_2(\psi_2(n, t)) \hat{H}_2(\psi_2(n, t)) = \hat{H}_2(\psi_1(m, t) \psi_2(n, t)) \)
\[ \Psi(m, n, t) \] behaves
as a constant, so far as \( \hat{H}_2 \) is concerned

[Note: \( \hat{H}_2(m_2) \) depends on \( m_2 \) only. Hence \( \hat{H}_2 \) depended on \( t \) as well, \( \Psi(m, n, t) \) could no more behave as a constant, because \( \hat{H}_1(m, t) \) has also \( t \)-dependence]
Similarly,
\[ \psi_2(\mathbf{r}+\mathbf{v}t) = \hat{H}_1 \left( \psi_1(\mathbf{r}+\mathbf{v} t) \psi_2(\mathbf{r}+\mathbf{v} t) \right) \]

In this case, \( \psi_2(\mathbf{r},t) \) behaves as a constant, so far as \( \hat{H}_1 \) is concerned.

Thus (iv) can be written as:
\[ \frac{\partial \psi_1}{\partial t} = \hat{H}_2(\mathbf{r}) \left( \psi_1(\mathbf{r},t) \psi_2(\mathbf{r},t) + \hat{H}_1(\mathbf{r}) \left( \psi_2(\mathbf{r},t) \psi_2(\mathbf{r},t) \right) \right) \]
\[ = \hat{H}_1(\mathbf{r}) \left( \psi_1(\mathbf{r},t) \psi_2(\mathbf{r},t) \right) \]
\[ = \hat{H} \psi \quad \text{[Since, } \psi(\mathbf{r},\mathbf{r},t) = \psi_1(\mathbf{r},t) \psi_2(\mathbf{r},t) \text{]}
\]
(Proved)

(b) We start with the core equation:
\[ \Delta \psi + \frac{\partial^2 \psi}{\partial t^2} = \hat{H} \psi \]

Where we need to check if \( \psi(\mathbf{r},\mathbf{r},t) \) can be of the form:
\[ \psi(\mathbf{r},\mathbf{r},t) = \psi_1(\mathbf{r},t) \psi_2(\mathbf{r},t) \]

Hence:
\[ \Delta \psi + \frac{\partial^2 \psi}{\partial t^2} = \hat{H} \psi \]

\[ = \left( \hat{H}_1 + \hat{H}_2 + \frac{\partial}{\partial t} \left( \hat{H}_1 + \hat{H}_2 \right) \right) \psi \]

\[ = \hat{H}_1(\mathbf{r}) \left( \psi_1(\mathbf{r},t) \psi_2(\mathbf{r},t) \right) \]

\[ = \hat{H} \psi \quad \text{[Since, } \psi(\mathbf{r},\mathbf{r},t) = \psi_1(\mathbf{r},t) \psi_2(\mathbf{r},t) \text{]}
\]

... (2)
\[ 
\psi_1(x, t) \psi_2(x', t) + \psi_1(x', t) \psi_2(x, t) 
\]

\[ 
= \left( \frac{\beta_1}{\beta_2} \right) \psi_1(x, t) \psi_2(x, t) + \psi_1(x, t) \left( \frac{\beta_2}{\beta_1} \psi_2(x', t) \right) 
\]

\[ 
= \frac{\beta_1 \beta_2}{\beta_1 + \beta_2} \psi_1(x, t) \psi_2(x, t) + \psi_1(x, t) \left( \frac{\beta_1 \beta_2}{\beta_1 + \beta_2} \psi_2(x', t) \right) 
\]

\[ 
= \frac{\beta_1 \beta_2}{\beta_1 + \beta_2} \psi_1(x, t) \psi_2(x, t) + \frac{\beta_1 \beta_2}{\beta_1 + \beta_2} \psi_1(x, t) \psi_2(x', t) 
\]

\[ 
= \frac{\beta_1 \beta_2}{\beta_1 + \beta_2} \left( \psi_1(x, t) \psi_2(x, t) + \psi_1(x, t) \psi_2(x', t) \right) 
\]

\[ 
\neq \text{L.H.S. (as found in (2))} 
\]

(5) According to Born's postulate, the wave function is directly related to the probability. 
Statistically, for two independent events, the joint probability \( P_{ij} \) can be written in the product form as follows:

\[ 
P_{ij} = P_i P_j \]

where \( P_i \) & \( P_j \) are individual probabilities of occurrences.

\( P_{ij} \) is a function of \( m_i \) & \( m_j \)

\[ 
P_{ij}(m_i, m_j) \] is a function of \( m_i \) only.

Similarly, \( P_i(m_j) \) is a function of \( m_j \) only.

Coming back to Born's postulate, the probability is the modulus squared of the wave function:

\[ 
P_{ij} = \left| \psi(m_i, m_j) \right|^2 
\]

\[ 
P_i = \left| \psi(m_i) \right|^2, \ P_j = \left| \psi(m_j) \right|^2 
\]
Now, \(|\psi(z, x)\rangle = |\psi(z)\rangle |\psi(x)\rangle \rangle^2\) if and only if \(\psi(z, x) = \psi(z) \psi(x)\).

Hence, for two independent events, the wave function of the composite event has to be the product of the wave functions of the individual events. Hence, wave equation that is 2nd order in time as in part (b), cannot be a valid equation for wave function, because the 2nd order wave equation is not necessarily satisfied by a wave function that can be split in product form.

**Prob 3.2**

We assume a special case, where \(\psi(z, t)\) can be written in the product form \(\psi(z, t) = \psi(z) \psi(t)\).

There is another assumption about \(\psi(z, t)\), which is:

it is an eigenstate of the Hamiltonian \(H\).

Hence, we have \(\hat{H} \psi = E \psi \) \(\tag{1}\)

and also from the equation \(\frac{\partial \psi}{\partial t} = \hat{H} \psi \) \(\tag{2}\)

From \(\ref{prob_3.2} \& \ref{prob_3.2}\) --

\[ \frac{\partial^2 \psi}{\partial t^2} = \frac{\partial^2 \psi}{\partial z^2} \] \(\tag{3}\)

\[ \int \frac{1}{\sqrt{\lambda}} \frac{d\phi}{dz} = \int \sqrt{\lambda} \phi(t) \] \(\tag{4}\)

\[ \int \frac{\partial \phi}{\partial t} = \int \phi(t) \] \(\tag{5}\)

Integrating, \(\phi = e^{Ez/\lambda}\)

Hence, \(\psi(z, t) = \psi(z) \ e^{Ez/\lambda}\) \(\tag{6}\)

Now, \(\psi(z, t)\) will be oscillatory in time if \(E/\lambda\) is a complex no.
But $E$ is real, since $E$ is an eigenvalue of the hamiltonian (or more intuitively, $E$ being energy must be real). Hence in order to $E \lambda$ be a complex no., $\lambda$ must be complex.

Proof 4.1.

\[ V(x) = 0 \quad -\pi \leq x \leq \pi \]

\[ = \infty \quad \text{otherwise} \]

\[ \psi(x) = \psi_0 (x) = 0 \quad (\text{since } V(x) \to \infty \text{ at } x \to \pm \pi) \]

The Schrodinger equation for $\psi_0$ is:

\[ -\frac{h^2}{2m} \frac{d^2 \psi_0}{dx^2} = E \psi_0 \]

\[ \Rightarrow \frac{d^2 \psi_0}{dx^2} = -\frac{2mE}{h^2} \psi_0 = -k^2 \psi_0 \]

\[ k = \frac{2mE}{h^2} \]

The solution of (i) is given as:

\[ \psi_0(x) = A_1 \sin kx + A_2 \cos kx \]

From (i),

\[ A_1 \sin k \pi + A_2 \cos k \pi = 0 \quad -(i) \]

\[ A_1 \sin \frac{k \pi}{2} + A_2 \cos \frac{k \pi}{2} = 0 \quad -(i) \]
Adding (iii) & (iv) we obtain —

\[ 2A_2 \cos \frac{ka}{2} = 0 \]

(iii) \(-\) (iv)

\[ -2A_1 \sin \frac{ka}{2} = 0 \]

So, we have,

\[ A_2 \cos \frac{ka}{2} = 0 \quad \text{—-(v)} \]

\[ A_1 \sin \frac{ka}{2} = 0 \quad \text{—-(vi)} \]

From (v) & (vi) we see there are two sets of possibilities —

\[ \cos \frac{ka}{2} = 0 \quad \text{and} \quad A_1 = 0 \quad \text{—-(vii)} \]

or,

\[ \sin \frac{ka}{2} = 0 \quad \text{and} \quad A_2 = 0 \quad \text{—-(viii)} \]

From (vii),

\[ \Psi_0 (n) = A_2 \cos \frac{ka}{2} \quad \text{where} \quad \cos \frac{ka}{2} = 0 \]

\[ \Rightarrow \frac{ka}{2} = (2n+1) \cdot \frac{\pi}{a} \]

\[ \Rightarrow k = (2n+1) \cdot \frac{\pi}{a} \]

From (viii),

\[ \Psi_1 (n) = A_1 \sin \frac{ka}{2} \quad \text{where} \quad \sin \frac{ka}{2} = 0 \]

\[ \Rightarrow \frac{ka}{2} = n \cdot \frac{\pi}{a} \]

\[ \Rightarrow k = 2n \cdot \frac{\pi}{a} \]

So, the solution is —

\[ \Psi_0 (n) = \text{constant} \cdot \cos \frac{ka}{2} \cdot n \quad \text{[when } b \text{ is even]} \]

\[ \Psi_1 (n) = \text{constant} \cdot \cos \frac{ka}{2} \cdot n \quad \text{[when } b \text{ is odd]} \]
The eigenenergy is \( \frac{\hbar^2}{2m} \alpha^2 \), as can be checked by using Schrödinger equation. We have

\[
-\frac{\hbar^2}{2m} a^2 = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial ^2} \left( \sin \left( \frac{\pi}{a} x \right) \right) = \left( -\frac{\hbar^2}{2m} \right) \left( \frac{\partial}{\partial x} \right) \left( \sin \left( \frac{\pi}{a} x \right) \right) = \left( -\frac{\hbar^2}{2m} \right) \left( \frac{\hbar^2}{2m} \frac{\partial}{\partial x} \right) \sin \left( \frac{\pi}{a} x \right)
\]

Hence the expression for eigenenergy for \( \alpha \) being even/odd is given by \( \frac{\hbar^2}{2ma} \). \( \alpha n \)

Prob 4.14

\[
\langle \bar{A} \rangle = \langle \bar{A} | A | \bar{A} \rangle = \langle \bar{A} | A | \bar{A} \rangle^* \quad \text{[since } A \text{ is Hermitian]}
\]

\[
= \langle A^+ | A | \bar{A} \rangle = \langle A^+ | A | \bar{A} \rangle^* \quad \text{[using the property of the inner product]}
\]

\[
= \langle A \rangle^* \quad \text{[since } \langle A \rangle = \langle A^+ | A | \bar{A} \rangle \]
\]

So, \( \langle \bar{A} \rangle \) is real. (Prove)
(a) The free-particle Hamiltonian is given as:

\[ H = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \]

\[ H \psi(x) = -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} \left( Ae^{ikx} + \frac{A}{\sqrt{2}} e^{-ikx} \right) \]

\[ = \left( -\frac{\hbar^2}{2m} \right) \left( (ik)^2 A e^{ikx} + \frac{A}{\sqrt{2}} (-ik) e^{-ikx} \right) \]

\[ = \left( \frac{\hbar^2 k^2}{2m} \right) \left( Ae^{ikx} + \frac{A}{\sqrt{2}} e^{-ikx} \right) \]

\[ = \frac{\hbar^2 k^2}{2m} \psi(x) \]

Yes, \( \psi(x) \) is an energy-eigenstate with the eigenvalue \( \frac{\hbar^2 k^2}{2m} \).

(b) The momentum operator is \(-i\hbar \frac{\partial}{\partial x}\).

\[ -i\hbar \frac{\partial}{\partial x} \psi(x) \]

\[ = -i\hbar \frac{\partial}{\partial x} \left( Ae^{ikx} + \frac{A}{\sqrt{2}} e^{-ikx} \right) \]

\[ = (-i\hbar) \left( (ik) Ae^{ikx} - ik \frac{A}{\sqrt{2}} e^{-ikx} \right) \]

\[ \neq \text{constant} \cdot \psi(x) \]

hence, \( \psi(x) \) is not an eigenstate of the momentum operator.
The possible values of momentum are 

\( \pm k \) and \( \mp k \).

\[ k \rightarrow \text{probability} = \frac{A^2}{A^2 + \left( \frac{4}{v_r^2} \right) \frac{1}{2}} = \frac{A^2}{A^2 + \frac{4}{v_r^2}} = \frac{A^2}{A^2 + \frac{4}{v_r^2}} = \frac{A^2}{2} \cdot \frac{2}{3} = \frac{2}{3} \]

\[ -k \rightarrow \text{probability} = \frac{\left( \frac{A^2}{v_r^2} \right)^2}{A^2 + \left( \frac{4}{v_r^2} \right) \frac{1}{2}} = \frac{A^2}{A^2 + \frac{4}{v_r^2}} = \frac{A^2}{2} \cdot \frac{2}{3} = \frac{1}{3} \]

Prob 9.17

\[ \hat{\psi}_1 (x) = \psi^* (x) \]

a) We need to show if \( \hat{C}^2 = \hat{C} \)?

\[ \langle \psi | \hat{C} | \psi \rangle = \int \psi^* (x) \hat{C} \psi (x) \, dx \]

\[ = \int \psi^* (x) \psi^* (x) \, dx = \left( \langle \psi | \psi \rangle \right)^* \]

\[ \langle \hat{C} \psi_1 | \psi_2 \rangle = \int \psi_1^* (x) \hat{C} \psi_2 (x) \, dx \]

\[ = \int \psi_1^* (x) \psi_1 (x) \psi_2 (x) \, dx \]

\[ = \langle \psi_1 | \psi_2 \rangle \]

Comparing (i) and (ii), we get:

\[ \langle \psi_1 | \hat{C} \psi_2 \rangle \neq \langle \hat{C} \psi_1 | \psi_2 \rangle \]

Hence \( \hat{C} \) is not hermitian.
To obtain eigenvalues & eigenfunctions we apply $\hat{c}^2$ twice.

The eigenvalue-eigenfunction definition for $\hat{c}^2$ is:

$$\hat{c}^2 \psi(x) = \lambda \psi(x) \quad \text{eigenfunction}$$

$$\text{eigenvalue} \quad \lambda \quad \text{-- (2)}$$

We apply $\hat{c}$ on both sides of (2):

$$\hat{c}(\hat{c}^2 \psi(x)) = \lambda (\hat{c} \psi(x))$$

or,

$$\hat{c} \psi(x) = \lambda (\hat{c} \psi(x))$$

Two cases:

(i) we have used (1)

(ii) we have not yet used the definition of $\hat{c}$

$$\Rightarrow \lambda^2 \psi(x) = \lambda \psi(x)$$

$$\Rightarrow \lambda = \pm 1 \quad \text{possible eigenvalues}$$

Case I: eigenvalue $\lambda = 1$

$$\hat{c}^2 \psi(x) = \lambda \psi(x)$$

$$= \psi(x)$$

$$\Rightarrow \psi(x) \text{ is a real function}$$

$$\text{So, eigenfunction is a real function}$$

Case II: eigenvalue $\lambda = -1$

$$\hat{c}^2 \psi(x) = \lambda \psi(x)$$

$$= -\psi(x)$$

$$\Rightarrow \psi(x) = -\psi(x)$$

$$\Rightarrow \psi(x) \text{ is a purely imaginary}$$

$$\text{function for eigenvalue -1.}$$