Prob 5.2

Normalization means we have to find \( A \), such that

\[
\int_0^\infty \psi(x,0)^* \psi(x,0) \, dx = 1
\]

\[
\Rightarrow (A^2) \int_0^\infty x^a (x-a)^a \, dx = 1
\]

or

\[
(A^2) \int_0^\infty x \cdot x^a - 2ax + a^a \, dx = 1
\]

or

\[
(A^2) \cdot \left[ \frac{x^3}{3} \right]_0^\infty - \frac{2a}{2} \cdot \frac{x^{a+1}}{a+1} \bigg|_0^\infty + \frac{a^a}{a} \cdot \left[ \frac{x^{a+1}}{a+1} \right]_0^\infty = 1
\]

or

\[
(A^2) \cdot \left[ \frac{a^a}{a} \cdot \left( \frac{5}{2} - \frac{a}{2} \cdot \frac{1}{a+1} \cdot \frac{a^a}{a} \cdot \xi^a \right) \right] = 1
\]

or

\[
(A^2) \cdot \left( \frac{5-15+10}{20} \right) = 1
\]

\[
A^2 = \frac{30}{a^5} \Rightarrow A = \sqrt{\frac{30}{a^5}}
\]

(b) The probability of finding particles in the interval \((0, a/2)\) is

\[
\int_0^{a/2} |\psi(x,0)|^2 \, dx
\]

We need to compute the integral. We can do it in a simpler way by symmetry arguments as follows —
\[ \psi(x,0) = x^2(\alpha - x)^2, \text{ where } \alpha = \sqrt{\frac{20}{\Delta}} \]

We plot \( \psi(x,0)^2 \) ---

\[ n \]

\( \psi(x,0)^2 \) is symmetrical about \( n = \alpha / 2 \). We can prove it as follows: We take \( n = \alpha / 2 + h \), hence

\[ \psi(x,0)^2 = \left( \alpha / 2 + h \right)^2 \left( \alpha / 2 - h \right)^2 = \alpha^2 \left( \alpha / 2 - h \right)^2 \left( \alpha / 2 - h \right)^2 \]  

Next we take \( n = \alpha / 2 - h \); the point \( n \) is taken to the left of \( \alpha / 2 \)

\[ \psi(x,0)^2 = \left( \alpha / 2 - h \right)^2 \left( \alpha / 2 - h \right)^2 = \alpha^2 \left( \alpha / 2 - h \right)^2 \left( \alpha / 2 + h \right)^2 \]

Comparing (i) and (ii), we get:

\[ \psi(x,0)^2 \bigg|_{n = \alpha / 2 + h} = \psi(x,0)^2 \bigg|_{n = \alpha / 2 - h} \text{ hence } \psi(x,0)^2 \text{ is symmetrical about the point } n = \alpha / 2. \]

Hence,

\[ \int_0^\alpha (\psi(x,0)^2) \, dx = \frac{1}{2} \int_0^\alpha (\psi(x,0)^2) \, dx \]

Thus, \( \int_0^\alpha \psi(x,0)^2 \, dx = \frac{1}{2} \int_0^\alpha (\psi(x,0)^2) \, dx \)

Hence, we have \( \alpha = 1 \)

Since it is proven, normality...
hence, no. of particles in this interval is:
\[ \text{no. of particles} \times \text{probability} = N \times \frac{1}{2} = \frac{N}{2} = \frac{500}{2} = 250 \]

(e) The energy eigenfunction corresponding to \( E_5 \) is:
\[ \psi_5 = \sqrt{\frac{2}{a}} \sin \frac{5\pi x}{a} \]

Thus, the probability of finding the particles in the above eigenstate is:
\[ P_5 = \left| \left\langle \psi_5 \right| \psi(n, 0) \right|^2 \]  \[ \| \psi_5 \| \| \psi(n, 0) \| \]

Now,
\[ \left\langle \psi_5 \right| \psi(n, 0) \right| = \int_0^a \sqrt{\frac{2}{a}} \sin \frac{5\pi x}{a} \cdot a (n-a) \, dx \]

One needs to compute this integral.

From (d), we get

Hence, no. of particles in the eigenstate \( \psi_5 \) is:
\[ N \cdot P_5 = (1000 \cdot P_5) \]  \[ \text{(and)} \]

(f) \[ \langle E \rangle = \frac{\hbar^2}{2m} \int_0^a \psi^* x \psi \, dx \], where \( \hbar = \frac{\pi \hbar}{2m} \) in the range \( 0 \leq x \leq a \)

\[ \langle E \rangle = \int_0^a (\hbar^2 \frac{x}{2m}) (\hbar^2 \frac{x}{2m}) \, dx \]  \[ \text{--- (g)} \]
\[ \frac{d^2}{dx^2} \left( x^n - x \right) = \frac{d}{dx} \left( x^n - x \right) = (2) \]

Hence, from (1), we get

\[ \langle E \rangle = \int_0^a \left( x^n - x \right) \, dx \]

\[ = \left[ \frac{x^{n+1}}{n+1} \right]_0^a - \left[ x \right]_0^a \]

\[ = \frac{a^{n+1}}{n+1} - a \]

\[ = -\frac{a^2 k^2}{m} \cdot \left[ \frac{a^3}{2} - a \cdot \frac{n^2}{2} \right] \]

\[ = -\frac{a^2 k^2}{m} \cdot \left[ \frac{a^5}{5} - \frac{a^3}{3} \right] \]

\[ = -\frac{a^2 k^2}{m} \cdot \frac{6a^2}{6} \]

\[ = \frac{5 a^2 k^2}{m a^2} \]

(Ans)

Prob 5.9

Measurement of position gives the value \( x = \frac{a}{2} \).

Eigenvalue of \( x \) is \( \frac{a}{2} \).

Eigenfunction of \( x \) is \( \delta \left( x - \frac{a}{2} \right) \).

(a) The energy eigenvalues are given as:

\[ \psi_n(x) = \sqrt{\frac{2}{a}} \sin \left( \frac{n\pi x}{a} \right), \quad n = 1, 2, 3, \ldots \]
After the position measurement, the wavefunction has collapsed to the position operator eigenstate, that is, 
\[ \delta (x - \frac{\pi}{2}) \].

Hence, the probability of finding a particle in odd-energy eigenstate is
\[ |\langle n | \delta (x - \frac{\pi}{2}) \rangle|^2 \], where \( n \) is odd.

\[ \begin{align*}
  &= \left| \int_{-\infty}^{\infty} \sqrt{\frac{\alpha}{2\pi}} \cdot \sin \frac{\psi \cdot x}{\alpha} \cdot \delta (x - \frac{\pi}{2}) \, dx \right|^2 \\
  &= \left| \sqrt{\frac{\alpha}{2\pi}} \cdot \sin \frac{\psi \cdot \frac{\pi}{2}}{\alpha} \cdot \frac{\pi}{2} \right|^2 \\
  &= \left| \sqrt{\frac{\alpha}{2\pi}} \cdot \sin \frac{n \pi}{2} \right|^2 \\
  &= \left( \frac{\alpha}{\pi} \right)^2 \\
  &\text{[using the fact that} \sin \frac{n \pi}{2} = 2, \text{ when} \ n \ \text{is odd]} \\
  &\text{do not depend on} \ n, \text{hence} \\
  &\text{the statement of equal probability.}
\end{align*} \]

(b) If \( n \) is even, all the above calculations done in (a) is valid up to the point: The probability of finding a particle in even-energy eigenstate is
\[ |\langle n | \delta (x - \frac{\pi}{2}) \rangle|^2 \]

\[ \begin{align*}
  &= \left| \int_{-\infty}^{\infty} \sqrt{\frac{\alpha}{2\pi}} \cdot \sin \frac{\psi \cdot x}{\alpha} \cdot \delta (x - \frac{\pi}{2}) \, dx \right|^2 \\
  &= \left| \sqrt{\frac{\alpha}{2\pi}} \cdot \sin \frac{\psi \cdot \frac{\pi}{2}}{\alpha} \cdot \frac{\pi}{2} \right|^2 \\
  &= \left| \sqrt{\frac{\alpha}{2\pi}} \cdot \sin \frac{n \pi}{2} \right|^2 \\
  &= \left( \frac{\alpha}{\pi} \right)^2 \\
  &= 0 \quad \text{[since} \ \sin \frac{n \pi}{2} = 0, \text{ when} \ n \ \text{is even]} \\
  &\text{(proven)}
\end{align*} \]
First we consider \( \psi = e^{i \lambda} \).

Momentum operator \( \hat{p} = -i \hbar \frac{d}{dx} \)

\[
\hat{p} \psi = -i \hbar \frac{d}{dx} (e^{i \lambda}) \\
= (-i \hbar \lambda) e^{i \lambda} \\
= \frac{\hbar}{2 \lambda} \psi
\]

hence, \( \psi \) is an eigenvector of momentum. Any
we would get the value \( \lambda \) with probability 1.

if we measure momentum.

Hamiltonian operator for a free particle is:

\[
\hat{H} = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2}
\]

\[
\hat{H} \psi = -\frac{\hbar^2}{2m} \frac{d^2}{dx^2} (e^{i \lambda}) \\
= \frac{\hbar^2 \lambda^2}{2m} \psi
\]

Hence, if energy is measured, we would get

\[
\frac{\hbar^2 \lambda^2}{2m}
\]

with probability 1.
Next, we consider \( \Psi = \beta \cos \alpha \)
\[
\Psi = \frac{\beta}{\sqrt{2}} (e^{i \alpha} + e^{-i \alpha})
\]
\[
\hat{\mathbf{p}} \Psi = \hat{\mathbf{p}} \frac{\beta}{\sqrt{2}} (e^{i \alpha} + e^{-i \alpha})
\]
\[
= \beta \frac{\hbar}{i} (e^{i \alpha} - e^{-i \alpha})
\]
\[
= \beta \hbar \left( \frac{e^{i \alpha} - e^{-i \alpha}}{2i} \right)
\]
\[
= \beta \hbar \cos \alpha
\]
\[
\hat{\mathbf{p}} \Psi = \beta \hbar \cos \alpha
\]
\[
\hat{\mathbf{p}} \Psi \neq \Psi
\]
momentum operator
\[
\hat{\mathbf{p}} \Psi = \beta \hbar \cos \alpha
\]
so, \( \Psi \) is not an eigenstate of \( \hat{\mathbf{p}} \). Hence
momentum measurement of \( \hat{\mathbf{p}} \) will yield
two possible values
\[
\pm \beta \hbar
\]
with equal probability.

We can figure out the above formula by using
the fact that \( e^{i \alpha} \) is a momentum eigenstate
corresponding to the eigenvalue \( \pm \beta \hbar \).
We proved it.
\( e^{i \alpha} \) is also a momentum eigenstate corresponding
to the eigenvalue \( \pm \beta \hbar \).

Next, energy measurement:
\[
\hat{\mathbf{p}} \Psi = \beta \hbar \cos \alpha
\]
\[
\hat{\mathbf{H}} \Psi = \frac{\hbar^2}{2m} \frac{d^2}{dx^2} \Psi
\]
\[
= \frac{\hbar^2}{2m} \frac{d^2}{dx^2} (\beta \cos \alpha)
\]
\[
= \frac{\beta^2 \hbar^2}{2m} (\beta \cos \alpha)
\]
\[
= \beta^2 \hbar^2 \cos \alpha
\]
\[
\hat{\mathbf{H}} \Psi = \beta^2 \hbar^2 \cos \alpha
\]
\[
\text{Energy eigenvalue, \( \beta^2 \hbar^2 \cos \alpha \) is eigenfunction.}
Since \( E_n \) is an eigenstate of the Hamiltonian, measurement of energy would yield the value \( \frac{\hbar^2 k^2}{2m} \) with probability 1.

One thing to note is: the momentum eigenstate given or \( \frac{\hbar^2 k^2}{2m} \) happens to be energy eigenstate too, hence if we measured momentum first, and then measured energy, we would still get the value \( \frac{\hbar^2 k^2}{2m} \) for energy.

**Prob. 5.3**

(a) \( \langle k \rangle = k \langle v \rangle \)

\[
\langle E \rangle = \sum \langle k \rangle \cdot E(k) = \sum k \cdot E(k) = \sum k \cdot \frac{\hbar^2 k^2}{2m}
\]

Thus, \( E_1 \) is the lowest eigenvalue where \( E_1 < E_n \) for \( n = 2, 3, \ldots \infty \)

From (i)

\[
\langle E \rangle = \sum \langle k \rangle \cdot E(k) = \sum k \cdot \frac{\hbar^2 k^2}{2m}
\]

\[
\leq \sum k \cdot \frac{\hbar^2 k^2}{2m} \cdot E_1 \quad \text{[we have replaced all \( E_n \)'s by \( E_1 \)]}
\]

\[
= E_1 \sum k \cdot \frac{\hbar^2 k^2}{2m} \cdot E_1 \quad \text{(we use \( \sum k \cdot \frac{\hbar^2 k^2}{2m} = \frac{\hbar^2}{2m} \) )}
\]

(b) The equality happens when \( \psi(x, t) \) is the energy eigenstate corresponding to the eigenvalue \( E_1 \).
Then all \( b_i \)'s are zero except \( n = 1 \), and \( L/E = E_i \).

**Problem 6.7**

\[ \psi(n) = \sin(kx) \]

**Why** \( \sin(kx) \) is an eigenstate of free-particle Hamiltonian \( H_0 = -\hbar^2/2m \cdot \frac{d^2}{dx^2} \)

Check:

\[ H_0 \psi(n) = -\frac{\hbar^2}{2m} \cdot \frac{d^2}{dx^2} (\sin(kx)) = \frac{\hbar^2}{2m} (k^2) \cdot (\sin(kx)) \]

Hence:

\[ \psi(n, t) = \psi(n) e^{-iE_k t/\hbar} = e^{-i\frac{\hbar k^2}{2m} t} \sin(kx) \]

\[ = e^{-i\frac{\hbar k^2}{2m} t} \frac{\hbar k}{m} \sin(kx) \]  

(6)

We have:

\[ \psi(n, t) = e^{-i\frac{\hbar k^2}{2m} t} \sin(kx) \]

We note that in the above expression for \( \psi(n, t) \), the momentum eigenfunction \( \sin(kx) \) either has eigenfunction coefficient \( e^{-i\frac{\hbar k^2}{2m} t} \) or \( e^{i\frac{\hbar k^2}{2m} t} \). The other eigenfunction coefficient corresponding to \( e^{-i\frac{\hbar k^2}{2m} t} \) is \( \frac{1}{\sqrt{2}} \), \( e^{-i\frac{\hbar k^2}{2m} t} \) and \( \frac{1}{\sqrt{2}} \).

Now:

\[ \left| \frac{1}{\sqrt{2}} e^{-i\frac{\hbar k^2}{2m} t} \right|^2 = \frac{1}{2} e^{-i\frac{\hbar k^2}{2m} t} \frac{1}{2} e^{i\frac{\hbar k^2}{2m} t} \]

Hence, at any time \( t \), the probability of
obtaining the values that are \( -\hbar k_0 \) in momentum measurement are equal, that is \( \frac{1}{2} \).

(c) After the momentum is surely known to be \( k_0 \), the wavefunction is surely known to be \( \psi_0 \).

\[ \psi_0 \text{ happens to be the eigenstate of the free-particle Hamiltonian } \hat{H}_0 \]

\[ \hat{H}_0 \psi_0 = \hbar^2 k_0^2 \psi_0 \].

We are interested in \( \psi_0 \) being an eigenstate of the Hamiltonian, because that knowledge is crucial in order to write the wave expression of \( \psi(\text{in}, t) \).

Now we can write:

\[ \psi(\text{in}, t) = e^{-i E \Delta t} \psi_0 \]

\[ E = \frac{\hbar^2 k_0^2}{2m} \]

where, \( \Delta t = t - t' \).
\[
\phi(x, t) = A e^{i(kx - \omega t)}
\]

We write the Schrödinger equation in its full form:
\[
\hat{H} \psi(x, t) = i\hbar \frac{\partial \psi(x, t)}{\partial t}
\]  
\[\text{(i)}\]

1.1.3
\[
\hat{H} \psi(x, t) = \left\{ -\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right\} A e^{i\alpha x} e^{-i\beta t}
\]
\[
= \left( -\frac{\hbar^2}{2m} \left( i\alpha \right)^2 + V(x) \right) A e^{i\alpha x} e^{-i\beta t}
\]
\[
= \left( \frac{\hbar^2 \alpha^2}{2m} + V(x) \right) A e^{i\alpha x} e^{-i\beta t} \quad \text{--- (ii)}
\]

1.1.5
\[
\psi(x, t) = A e^{i\alpha x} e^{-i\beta t}
\]
\[
\text{ib} \frac{\partial \psi}{\partial t} = (\text{ib}) \left( A e^{i\alpha x} \right) \cdot e^{-i\beta t}
\]
\[
= \text{ib} A e^{i\alpha x} e^{-i\beta t} \quad \text{--- (iii)}
\]

Using (ii) & (iii) in (i), we obtain:
\[
\left( \frac{\hbar^2 \alpha^2}{2m} + V(x) \right) A e^{i\alpha x} e^{-i\beta t} = \text{ib} A e^{i\alpha x} e^{-i\beta t}
\]

Hence, we obtain:
\[
\frac{\hbar^2 \alpha^2}{2m} + V(x) = \text{ib}
\]
\[
V(x) = \text{ib} - \frac{\hbar^2 \alpha^2}{2m}
\]

\[\alpha \text{ is a constant (say } V_0)\]
(a) \( v = V_0 = \text{constant} \)

(b) \[
\hat{p} (e^{iax} e^{-ibt}) \\
= -i\hbar \frac{\partial}{\partial x} (e^{iax} e^{-ibt}) \\
= (-i\hbar) \cdot (ia) \cdot e^{iax} e^{-ibt} \\
= (i\hbar a) \cdot e^{iax} e^{-ibt} \\
\]

Since \( \psi(x,t) \) is an eigenfunction of \( \hat{p} \), we could obtain the eigenvalue \( i\hbar a \) in case of momentum measurement.

(c) We use the alternative form of energy operator which is \( i\hbar \frac{\partial}{\partial t} \). That this is an energy operator follows from Schrödinger equation \( \hat{H}\psi = i\hbar \frac{\partial}{\partial t} \psi \). Schrödinger equation makes \( \hat{H} \) and \( i\hbar \frac{\partial}{\partial t} \) the same thing.

Now, \[
\begin{align*}
(i\hbar) \cdot (a e^{iax}) \\
= (i\hbar a) \cdot e^{iax} \\
\end{align*}
\]

We see, \( \psi(x,t) \) is an eigenstate of the energy.
operator, too. Hence energy measurement when the state \( \psi_{\text{min}} \) is \( A \exp[-i(t\lambda - b)] \)
would give the value \( 2b \).

\[ \text{Problem 9.25} \]

(a) \[ \hat{L} = \text{possible values} \]
\[ \begin{align*}
\langle \hat{L} \rangle & \to \text{probability} = \left( \frac{3}{\sqrt{6}} \right)^2 = \frac{9}{26} \\
\langle \hat{L} \rangle & \to \text{probability} = \left( \frac{4}{\sqrt{26}} \right)^2 + \left( \frac{1}{\sqrt{26}} \right)^2 \\
& = \frac{16}{26} + \frac{1}{26} \\
& = \frac{17}{26}
\end{align*} \]

(b) \[ \hat{L}_z \]
\[ \begin{align*}
\langle \hat{L}_z \rangle & \to \text{probability} = \left( \frac{3}{\sqrt{6}} \right)^2 + \left( \frac{1}{\sqrt{26}} \right)^2 = \frac{10}{26} \\
\langle \hat{L}_z \rangle & \to \text{probability} = \left( \frac{4}{\sqrt{26}} \right)^2 = \frac{16}{26}
\end{align*} \]

(b) Hamiltonian is given by \[ \frac{\hat{L}_z^2}{2I} \]

Hence eigenvalues of the Hamiltonian are \[ \frac{2h^2}{2I} \text{ and } \frac{52h^2}{2I} \] we have used the fact that the eigenvalues of \( \hat{L}_z \) are \( \pm \frac{\hat{L}_z}{\sqrt{2I}} \) respectively.
hence,
\[ Y(\theta, \phi, \lambda) = \frac{3}{\sqrt{26}} Y_1^1 e^{-i \frac{5\lambda^2}{2} t} + \frac{1}{\sqrt{26}} Y_{11} e^{-i \frac{5\lambda^2}{2} t} + \frac{1}{\sqrt{26}} Y_{10} e^{-i \frac{5\lambda^2}{2} t} \]

\[ = \frac{2}{\sqrt{26}} Y_1^1 e^{-i \frac{5\lambda^2}{2} t} + \frac{1}{\sqrt{26}} Y_{11} e^{-i \frac{5\lambda^2}{2} t} + \frac{1}{\sqrt{26}} Y_{10} e^{-i \frac{5\lambda^2}{2} t} \]

\[
\langle E \rangle = \left\langle \frac{2\nu}{\nu_s} \right\rangle
\]
\[ = \frac{1}{2\nu_s} \left[ \langle 2\nu^2 \rangle \right] \]
\[ = \frac{1}{2\nu_s} \left[ \frac{9}{26} + \left( \frac{56\nu^2}{26} \right) \right] \]

We have used the fact that the eigenvalues of \( \nu^2 \) are
\[ 2\nu^2 \rightarrow \text{probability} \frac{9}{26} \]
\[ 56\nu^2 \rightarrow \text{probability} \frac{17}{26} \]

\[ \frac{2\nu}{\nu_s} = \frac{\nu^2}{\nu_s} \left[ \frac{9}{26} + \frac{2\times17}{26} \right] \]
\[ = \frac{\nu^2}{\nu_s} \left[ \frac{9}{26} + \frac{34}{26} \right] \]
\[ = \frac{\nu^2}{\nu_s} \left[ \frac{43}{26} \right] \]

(Ans.)

Now put numerical values for \( \nu \) & \( \nu_s \).
Prob 9. 2.6

\( y(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cdot \sin \theta \sin \phi \)

We see from the table that:

\[
Y_{1}^1 + Y_{1}^{-1} = \frac{1}{\sqrt{2}} \left( \frac{3}{4\pi} \right)^{1/2} \left[ -\sin \theta e^{i\phi} + \sin \theta e^{-i\phi} \right]
\]

\[
= \frac{1}{\sqrt{2}} \left( \frac{3}{4\pi} \right)^{1/2} \sin \theta (-i \sin \phi)
\]

hence from (i) we get:

\[
\sin \theta \sin \phi = \frac{Y_{1}^1 + Y_{1}^{-1}}{(\frac{3}{4\pi})^{1/2} \sin \theta}
\]

hence,

\[
y(\theta, \phi) = \sqrt{\frac{3}{4\pi}} \cdot \sin \theta \sin \phi
\]

\[
= \frac{1}{\sqrt{2}} \left( \frac{3}{4\pi} \right)^{1/2} \left( Y_{1}^1 + Y_{1}^{-1} \right)
\]

\[
= \frac{1}{\sqrt{2}} \left( Y_{1}^1 + Y_{1}^{-1} \right)
\]

\[
= \frac{1}{\sqrt{2}} Y_{1}^1 + \frac{i}{\sqrt{2}} Y_{1}^{-1}
\]

(2) From (i) we can write:

\[
L \rightarrow \hat{L} \rightarrow \text{probability } \frac{1}{2}
\]

\[
-\hat{L} \rightarrow \text{probability } \frac{1}{2}
\]

[We have used the fact that \( Y_{1} \) are eigenfunctions of \( \hat{L} \).]
We know,
\[ \hat{\mathbf{n}} = \hat{\mathbf{e}} \left( \sin \theta \frac{\partial}{\partial \phi} + \cot \theta \cos \phi \frac{\partial}{\partial \theta} \right) \]
\[ \langle \hat{\mathbf{n}} \rangle = \int \psi^*(r, \theta, \phi) \hat{\mathbf{n}} \psi(r, \theta, \phi) \sin \theta \, d\theta \, d\phi \quad \text{--- (iii)} \]

Note: This is not simply \( \hat{\mathbf{e}} \, \sin \theta \, d\phi \), but \( \sin \theta \, d\theta \, d\phi \); the factor \( \sin \theta \) appears because the integration is a 3-dimensional polar integration.

Now,
\[ \langle \hat{\mathbf{n}} \rangle = \int \psi^*(r, \theta, \phi) \left( \sin \theta \frac{\partial}{\partial \phi} \psi(r, \theta, \phi) + \cot \theta \cos \phi \frac{\partial}{\partial \theta} \psi(r, \theta, \phi) \right) \sin \theta \, d\theta \, d\phi \]
\[ = \int \psi^* \left( \sin \theta \frac{\partial}{\partial \phi} \psi + \cot \theta \frac{\cos \theta}{\frac{\partial}{\partial \theta} \psi} \right) \sin \theta \, d\theta \, d\phi \]
\[ = \int \psi^* \left( \frac{\partial}{\partial \phi} \sin \theta \cos \phi \frac{\partial}{\partial \theta} \psi + \cot \theta \frac{\cos \theta}{\frac{\partial}{\partial \theta} \psi} \right) \sin \theta \, d\theta \, d\phi \]
\[ = \int \psi^* \frac{\partial}{\partial \phi} \sin \theta \cos \phi \frac{\partial}{\partial \theta} \psi + \cot \theta \frac{\cos \theta}{\frac{\partial}{\partial \theta} \psi} \sin \theta \, d\theta \, d\phi \]
\[ \text{Linearity, from (i) ---} \]
\[ \langle \hat{\mathbf{n}} \rangle = \int \int \psi^* \left( \frac{\partial}{\partial \phi} \psi_{\theta} + \cot \theta \frac{\cos \theta}{\frac{\partial}{\partial \theta} \psi} \right) \sin \theta \, d\theta \, d\phi \]
\[ = \int \left( \frac{\partial}{\partial \phi} \psi_{\theta} \right) \left( \int \psi \sin \theta \, d\theta \right) \frac{\partial}{\partial \theta} \psi + \cot \theta \frac{\cos \theta}{\frac{\partial}{\partial \theta} \psi} \sin \theta \, d\theta \, d\phi \]
\[ = \int \psi \frac{\partial}{\partial \phi} \left( \sin \theta \cos \phi \frac{\partial}{\partial \theta} \psi \right) \sin \theta \, d\theta \, d\phi \]
\[ \text{--- (iv)} \]
Now, \[ \int_{0}^{2\pi} \sin \theta \, d\theta = 0 \]

Thus, (iv) is satisfied. That is \( \langle \hat{\alpha}^* \rangle = 0 \) (Ans).

(5) Both \( \gamma_1 \) and \( \gamma_{-1} \) are eigenstates of the operator \( \hat{\gamma} \) with the eigenvalue \( \gamma^2 (1) (1 + 1) = 2 \). Hence, for \( \psi(0, \theta) = \frac{1}{\sqrt{2}} \gamma_1 + \frac{i}{\sqrt{2}} \gamma_{-1} \) (using (ii)),

\[ \langle \hat{\gamma}^* \rangle = \langle \hat{\gamma} \rangle = \left\{ \frac{\gamma^2 (1 + 1)}{2 \sqrt{2} \left( \frac{1}{2} + \frac{1}{2} \right)} \right\} = \left\{ \frac{2}{4} \right\} = \left\{ \frac{1}{2} \right\} \] (Ans).