Physics 263

Deriving the Fresnel Equations

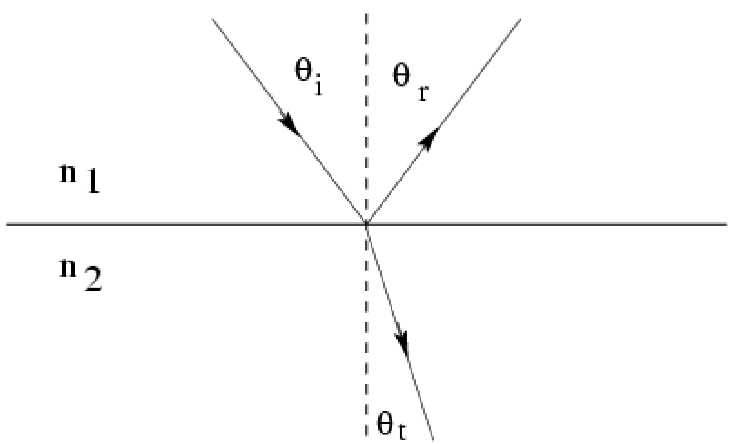
1 Introduction

The intensity of light reflected from the surface of a dielectric, as a function of the angle of incidence was first obtained by Fresnel in 1827.

When an electromagnetic wave strikes the surface of a dielectric, both reflected and refracted waves are generally produced. The reflected wave has a direction given by the "Law of Reflection"[1] :

$$\theta_r = \theta_i$$

where the angles are between the rays and a line perpendicular to the reflecting surface.



Electromagnetic theory predicts the ratio of the intensity of the reflected light to the intensity of the incident light. The polarization of the light with respect to the plane of reflection [2] must be taken into account. There are two extreme cases: (1) the electric field is perpendicular to the plane of reflection, called **Transverse Electric**, and (2) the magnetic field is perpendicular to the plane of reflection, **Transverse Magnetic**.

[1] The "Laws" of reflection and refraction are actually theorems which can be derived from electromagnetic theory

[2] The plane of reflection is the plane defined by the incident and reflected rays.

These two cases are illustrated in the figure below:

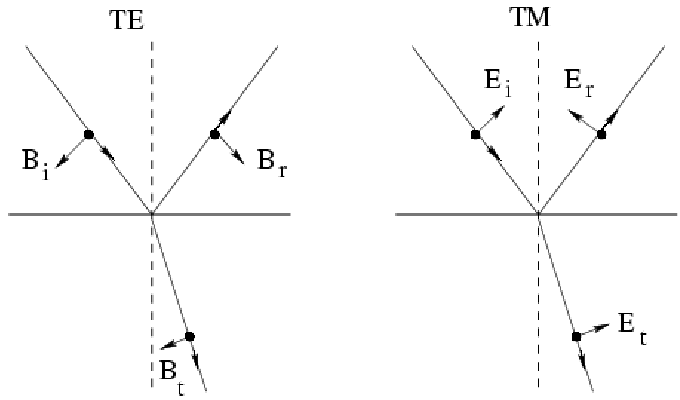


Figure 1: TE and TM reflections. The dot shows the direction of the electric field in the TE case, and the magnetic field in the TM case.

2 TE Equation

The laws of electromagnetism, applied to this case, give the following boundary conditions:

1. The perpendicular component of B is continuous across the boundary between the two media.
2. The parallel component of E is continuous across the boundary between the two media.

In this case, since $\mathbf{E}$ is parallel to the boundary,

$$E_i + E_r = E_t$$

The continuity of the perpendicular components of B is expressed by

$$B_i \cos\theta_i - B_r \cos\theta_r = B_t \cos\theta_t$$

To eliminate B, we use the relations

$$\frac{E}{B} = c = \frac{\omega}{k} \qquad so \quad B = \frac{k}{\omega} E$$

Now ω is the same in both media, so

$$k_1(E_i - E_r)\cos\theta_i = k_2 E_t \cos\theta_t$$

where we have used the law of reflection: $\theta_i = \theta_r$, The index of refraction, n is generally

$$n = \frac{c}{v} = \frac{c}{\frac{\omega}{k}} = \frac{k}{\omega}c$$

So, cancelling ω and c,

$$n_1(E_i - E_r)\cos\theta_i = n_2 E_t \cos\theta_t$$

Substituting for E_t gives

$$E_i(n_1\cos\theta_i - n_2\cos\theta_t) = E_r(n_1\cos\theta_i + n_2\cos\theta_t)$$

So the ratio of reflected to incident amplitudes is

$$\frac{E_r}{E_i} = \frac{(n_1\cos\theta_i - n_2\cos\theta_t)}{(n_1\cos\theta_i + n_2\cos\theta_t)}$$

Now let $n = n_2/n_1$, then the above equation becomes

$$\frac{E_r}{E_i} = \frac{(\cos\theta_i - n\cos\theta_t)}{(\cos\theta_i + n\cos\theta_t)}$$

Finally we use the law of refraction: $n_1 \sin\theta_i = n_2 \sin\theta_t$ to write

$$\cos\theta_t = \sqrt{1 - \sin{\theta_t}^2} = \sqrt{1 - \frac{\sin^2\theta_i}{n^2}}$$

And so,

$$\frac{E_r}{E_i} = \left(\frac{\cos\theta - \sqrt{n^2 - \sin^2\theta}}{\cos\theta + \sqrt{n^2 - \sin^2\theta}}\right)$$

The intensity ratio is then

$$R_{TE} = \frac{I_r}{I_i} = \left(\frac{\cos\theta - \sqrt{n^2 - \sin^2\theta}}{\cos\theta + \sqrt{n^2 - \sin^2\theta}}\right)^2 \quad (1)$$

3 TM Equation

For the transverse magnetic case, using similar methods, the result is

$$R_{TM} = \frac{I_r}{I_i} = \left(\frac{-n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}}{n^2\cos\theta + \sqrt{n^2 - \sin^2\theta}}\right)^2 \quad (2)$$