I. HARPER’S EQUATION DESCRIBING ELECTRONS IN A CRYSTAL IN A MAGNETIC FIELD

In the tight binding approximation, the eigenstates for an electron in a square lattice with flux $\phi = B a^2 / \phi_0$ are given by Harper’s equation,

$$\psi_{m+1} + \psi_{m-1} + 2 \cos(2\pi \phi m - k_y) \psi_m = E \psi_m$$  \hspace{1cm} (1)

This equation was derived in 1955 by Philip Harper and is known as the Harper’s equation. Note, again, the two-dimensional problem is reduced to one-dimensional problem.

**Important parameter $\phi$**

The parameter $\phi$ that appears in Harper equation is the *pure number* $\phi$ (“pure” in the sense that it has no units attached to it), which measures the magnetic field’s strength in an extremely natural fashion. This pure number, with no units at all, tell us how strong a magnetic field is. It all hinges on the fact that there is a fundamental amount of magnetic flux — the *flux quantum*,
equal to $\hbar c/e$ — that emerges intrinsically out of quantum mechanics. This minuscule quantity is an inherent fact about our universe, just as are the speed of light and the charge on the electron. Given that this tiny amount of flux is the natural chunk of flux, it is as if nature had handed us a measuring stick on a silver platter! This beautiful and generous favor on nature’s part must not be ignored.

$$
\begin{pmatrix}
2C_1 & 1 & 0 & 0 & \cdots & e^{-ik_x} \\
1 & 2C_2 & 1 & 0 & 0 & \cdots \\
0 & 1 & 2C_3 & 1 & 0 & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
e^{ik_x} & 0 & 0 & 0 & 1 & 2C_q
\end{pmatrix}
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\vdots \\
\psi_q
\end{pmatrix}
= E
\begin{pmatrix}
\psi_1 \\
\psi_2 \\
\psi_3 \\
\vdots \\
\psi_q
\end{pmatrix}
$$

where $C_n = \cos(2\pi n\phi - k_y)$. The lower-left and upper-right corner terms $e^{\pm ik_x}$ in this matrix reflect Bloch’s theorem, which assumes periodic boundary conditions — namely, $\psi_n = e^{ik_x}\psi_{n+q}$. In a handful of very simple cases, such as $\phi = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, the eigenvalues $E$ can be determined analytically. However, for $q > 4$, the above matrix-eigenvalue problem can only be solved numerically.

**Rational vs Irrational Flux**

Let $\phi = \frac{p}{q}$.

For rational flux, Harper equation describes a system that is periodic with period $q$. Such systems can be studied using “Bloch Theorem”.

**Bloch’s theorem**

Named after physicist Felix Bloch, Bloch’s theorem states that the energy eigenstates of an electron moving in a crystal (a periodic potential) can be written in the following form:

$$
\Psi(r) = e^{ik\cdot r} u(r)
$$

where $u(r)$ is a periodic function with the same periodicity as that of the underlying potential
— that is, $u(r) = u(r + a)$. The exponential preceding the periodic function $u$ is a kind of helical wave, or “corkscrew”, which multiplies the wave function by a spatially-changing phase that twists cyclically as one moves through space in a straight line.

*Some Analytic Results*

(1) Analytic expressions for the energy dispersions $E(k_x, k_y)$ for a few simple cases

(a) For $\phi = 1$, the energy spectrum consists of a single band given by $E = 2(\cos k_x + \cos k_y)$.

(b) For $\phi = 1/2$, the two bands having energies $E_+$ and $E_-$ are given by $E_{\pm} = \pm 2\sqrt{\cos^2 k_x + \cos^2 k_y}$.

(c) For $\phi = 1/3$, the three bands having energies $E_0$, $E_1$, and $E_2$ are given by $E_i = 2\sqrt{2}\cos(\theta \pm i\frac{2}{3}\pi)$. Here, $\theta = \frac{1}{3}\arccos\left[(\cos 3k_x + \cos 3k_y)/2\sqrt{2}\right]$.

(d) For $\phi = 1/4$, the four bands having energies $E_{++}$, $E_{+-}$, $E_{-+}$, and $E_{--}$ are given by the expression $E_{\pm \pm} = \pm \sqrt{4 \pm 2\left[3 + \frac{1}{2}(\cos 4k_x + \cos 4k_y)\right]^2}$.
• Using analytical expressions in four cases, understand what is meant by a band.

• For every rational flux \( p/q \), there are \( q \) bands. In other words, a single band for zero magnetic field splits into \( q \) bands when magnetic flux is a rational number with denominator \( q \).

• For \( q \)-even, two bands touch at \( E=0 \).

• For irrational value of \( \phi \), we have infinity of bands – known as a Cantor set (A type of fractal). It turns out that for each irrational, the sum of the band widths is zero, that is, the Cantor set has a zero measure. The proof that the set forms a Cantor set has a long history and is known as Ten Martini Problem.

• Smooth gaps in otherwise fractal structure.

• As we will discuss later, the integers are the Berry phases in units of \( 2\pi \). That is the anholonomy or total curvature as we integrate over \((k_x, k_y)\) plane – the Brillouin zone. It turns out that these integers are the quantum numbers of Hall conductivity.

Recall: the concept of Brillouin zone

This notion was developed by the French physicist Léon Brillouin (1889–1969). For any crystal lattice in three-dimensional physical space, there is a “dual lattice” called the reciprocal lattice, which exists in an abstract space whose three dimensions are inverse lengths. This space lends itself extremely naturally to the analysis of phenomena involving wave vectors (because their dimensions are inverse lengths). If we limit ourselves to crystals whose lattices are perfectly rectangular (as has generally been done in this book), then given a lattice whose unit cell has dimensions \( a \times b \times c \), the reciprocal lattice’s unit cell will have dimensions \( \frac{1}{a} \times \frac{1}{b} \times \frac{1}{c} \). This cell is called the first Brillouin zone. The various locations in the Brillouin zone — wave vectors — act as indices labeling the different Bloch states.

In two-dimension, the Brillouin zone is a torus.
II. SIMPLEST MODEL OF QUANTUM HALL: FLUX \(-\frac{1}{2}\)

A square lattice immersed in a magnetic field, where electrons can only tunnel to its neighboring sites is described by Harper’s equation:

\[
\psi_{m+1} + \psi_{m-1} + 2 \cos(2\pi \phi m - k_y) \psi_m = E \psi_m
\] (3)

In the above model, all rational flux-values except for \(\phi = \frac{1}{2}\) support a quantum Hall state due to the presence of a gap.

However, flux-\(\frac{1}{2}\) is the simplest case as the Hamiltonian is a \(2 \times 2\) matrix and hence describes a spin-\(\frac{1}{2}\) model in a magnetic field.

Mapping to Spin-\(\frac{1}{2}\) in a Magnetic Field

Let us denote hopping along \(x\) and \(y\) directions as \(J_x\) and \(J_y\). For the square lattice described above, we have set \(J_x = J_y = 1\).

\[
H(k_x, k_y) = -\vec{h} \cdot \vec{\sigma},
\] (4)

\[h_x = 2J_x \cos k_x; \quad h_y = 2J_y \cos k_y; \quad h_z = 0\] (5)

With \(h_z = 0\), we have \(\theta = 0\). This means there is no Berry phase in the problem.

Generalized Harper Model

Consider a generalization of the above model where electrons can also tunnel along the diagonal. This opens a gap at \(\phi = \frac{1}{2}\) as shown in Fig. (2). This also maps to a spin in a magnetic field problem.

This simple case of flux-value \(\frac{1}{2}\), a generalization of Harper’s equation in which the crystal electrons can hop along diagonals (in addition to nearest-neighbor hopping) turns out to be the ideal model to describe a quantum Hall system. Such a system maps exactly onto a spin-\(-\frac{1}{2}\) situation in a magnetic field, as described above. With the diagonal hopping, which we will denote by \(J_d\), the Hamiltonian of this system can be transformed as follows:
\[ H(k_x, k_y) = -\vec{h} \cdot \vec{\sigma}, \]  
\[ h_x = 2J_x \cos k_x; \quad h_y = 2J_y \cos k_y; \quad h_z = 4J_d \sin k_x \sin k_y \]  

Unlike the spin-$\frac{1}{2}$ situation described above, where the parameter space can be realized as a sphere, the parameter space here is the space of Bloch wave-number vectors \((k_x, k_y)\). This space — a Brillouin zone — is not a sphere but a torus. If we are dealing with a rational magnetic flux-value \(p/q\), then the edges of the Brillouin zone will be defined by \(k_y\) lying within the interval \([-\pi/q, +\pi/q]\) on one axis (as can be seen from the periodicity of the cosine in Harper equation), and \(k_x\) lying within the interval \([-\pi, +\pi]\) along the perpendicular axis.

Although the problem is thereby mapped to a spin-$\frac{1}{2}$ problem in a magnetic field, we note that there is no point in the Brillouin zone where \(h = 0\) (which is the location of the monopole). Therefore, the fictitious monopole in the quantum Hall problem exists outside the reciprocal space. It can be seen if we consider samples with edges where from the crossing-point of the two energy eigenvalues of the edge modes, we can pinpoint the monopole inside the gap.

**FIG. 2:** Butterfly plot when electrons are allowed to tunnel to the diagonal.