Week 8 – Oct 17 & Oct 19, 2016

Projects

(1) Matter-Antimatter

Resource: Lecture by Maurice Jacob: PaulDirac.pdf and recent <https://en.wikipedia.org/wiki/Antimatter>

(2) Majorana Fermions

Nature Physics Article by Frank Wilczek: Majorana Reurns.

(3) Topological Quantum Computing

Topological Quantum Computing by Sankar Das Sarma, Michael Freedman, and Chetan Nayak

---

I. BERRY MAGNETISM AND MAGNETIC MONOPOLE

$$\gamma \ = \ i \oint < \psi_n | \nabla_{\vec{R}} | \psi_n > \cdot d\vec{R} \tag{1}$$

Remarkably, one can identify an "effective electromagnetism" lurking in this situation by thinking of $< \psi | \nabla_{\vec{R}} | \psi >$ as an effective magnetic vector potential, which we will denote by $\vec{A}_{\text{eff}}$:

$$\vec{A}_{\text{eff}} = i < \psi | \nabla_{\vec{R}} | \psi >, \ \gamma \ = \ \oint \vec{A}_{\text{eff}} \cdot d\vec{R} \tag{2}$$

Here, $\nabla_{\vec{R}}$ is the gradient operator in the abstract parameter space. The parameters collected in the vector $\vec{R}$ might, for instance, represent angles θ and ϕ, if the parameter space is, say, the surface of a sphere.

$$\gamma = \int_S \nabla_R \times \vec{A}_{\text{eff}} \cdot d\vec{S} = \int_S \vec{B}_{\text{eff}} \cdot d\vec{S} \tag{3}$$

One can easily check that $\vec{A}_{\text{eff}}$ has the following property,

$$\psi \to e^{i\beta}\psi, \quad \text{then} \quad \vec{A}_{\text{eff}} \to \vec{A}_{\text{eff}} - \vec{\nabla}_R \beta. \tag{4}$$

A. Gauge Theories

A gauge field theory is a special type of theory, in which matter fields (like electrons and quarks, which make up protons and neutrons) interact with each other via forces that are mediated by the exchange of vector bosons (like photons and gluons, which bind quarks together in nucleons). Electromagnetic theory described by Maxwell's equations is the simplest kind of gauge theory, known as the $U(1)$ gauge theory, as the gauge transformation (unitary transformation) involves a simple one-component scalar β. It turns out that in other cases of quarks etc, β is a matrix.

II. CALCULATING GEOMETRIC PHASE – EXAMPLES

(I) Spin in a Magnetic field of constant Magnitude but changing direction.

$$H = -\mu_s \cdot B = g_s \mu_b \frac{\vec{S} \cdot \vec{B}}{\hbar} \tag{5}$$

where $\mu_B = \frac{e\hbar}{2m}$ and $\vec{S} = \frac{\hbar}{2}\hat{\sigma}$.

$$\begin{aligned} H &= -\mu_b \vec{B} \cdot \vec{\sigma} = -\mu_b B_0 \hat{n} \cdot \vec{\sigma} \\ &= -\mu_b B_0 \begin{pmatrix} \cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & -\cos\theta \end{pmatrix} \end{aligned}$$

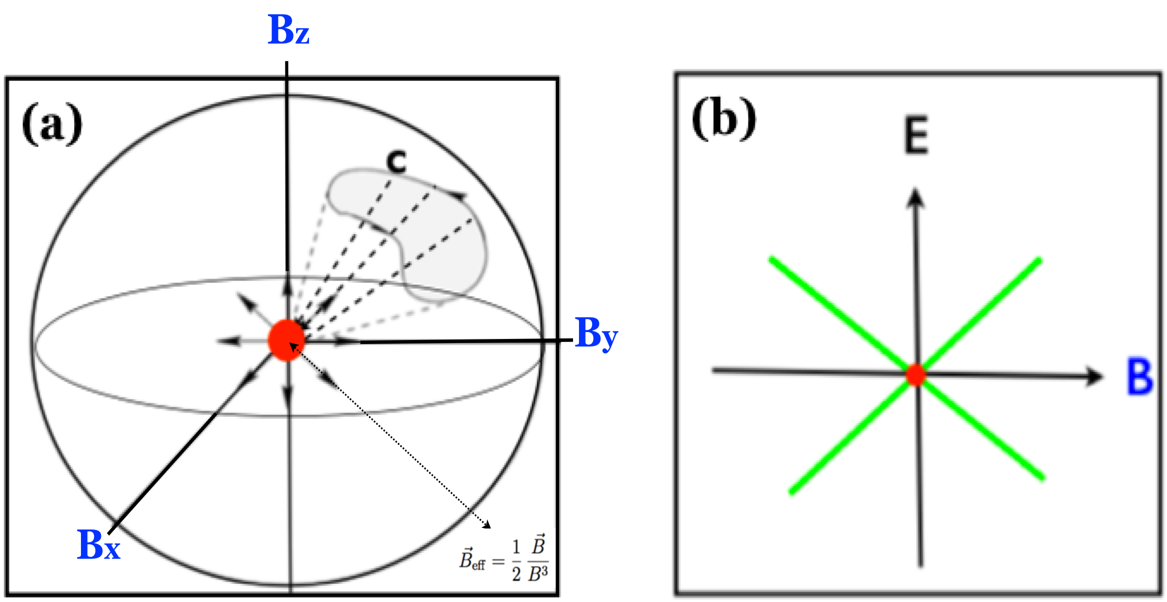

FIG. 1: The curve C denotes the path of the magnetic field. It is very important to distinguish $\vec{B}$ and $\vec{B_{eff}}$.

where B_0 is the magnitude of the $\vec{B}$ and $\hat{n}$ is a unit vector pointing in the direction of the magnetic field: $\vec{B} = B_0\hat{n}$. We have used the expression for Pauli matrices to write the above matrix form of H.

The eigenvalues are given by,

$$E_{\pm} = \pm\mu_b B_0 \tag{6}$$

. It will suffice for us to consider just one of the eigenstates of the system, such as this:

$$\Psi_+ = \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{+i\phi}\sin\frac{\theta}{2} \end{pmatrix} \tag{7}$$

The parameter vector $R(t)$ described above is specified by $(\theta(t), \phi(t))$. We now use the formula $\gamma = i\oint <\psi_n|\nabla_{\vec{R}}|\psi_n> \cdot d\vec{R}$ to calculate the geometric phase γ.

$$
\begin{aligned}
\vec{B}_{eff} &= \frac{\hat{B}}{2B_0^2} \\
&\equiv \frac{\hat{r}}{2r^2}
\end{aligned}
$$

Therefore, the effective magnetic field is the field of the monopole sitting at the center of the sphere. The geometric phase is the flux enclosed by the path of the applied magnetic field due to an imaginary monopole that sits at the center of the sphere.

The geometric phase $\gamma = \int \frac{i}{2r^2}\hat{r} \cdot da = -\frac{1}{2}\Omega$, where Ω is the solid angle.

A. Monopoles

From the formula for anholonomy – the geometric phase, emerges an elegant theoretical description that deepens our understanding of these phenomena.

Physicists have a deep drive to seek simple ways to picture anything new they encounter. They pursue this goal with great alacrity, driven by their natural faith that all physical phenomena, when viewed from the right perspective, are connected in some way. Their journey usually begins with questions like "Where does this phenomenon belong within the framework of familiar equations or familiar laws of physics?" or "To what familiar physical phenomenon might this new phenomenon be analogous?" A sense of simplicity and elegance plays a central role in reincarnating, in exotic new contexts, mathematical ideas that originally sprang up in completely different contexts.

Since anholonomic processes involve phase shifts, it is intuitively plausible that such processes might have some deep analogical links to Maxwell's equations. Indeed, readers familiar with particle physics will recall that phases are associated with gauge fields, and Maxwell's equations are perhaps the simplest form of gauge fields known to physicists. In fact, the gauge invariance of Maxwell's equations is the core reason underlying the fundamental law of conservation of charge.

As it turns out, this intuitive guess that there might be an analogical link connecting electromagnetic phenomena to anholonomy is deeply correct.This fictitious magnetic field, along with its corresponding vector potential (the "Berry connection"), forms the crux of the mathematics of anholonomic processes. This radically novel way of describing anholonomy, often called “Berry magnetism”, has many striking features, some of which are *disanalogies* with genuine electromagnetism. For instance, unlike the actual equations of Maxwell, which have no solutions involving magnetic monopoles, Berry magnetism permits the existence of (effective) magnetic monopoles.

Equation (4) is strongly reminiscent of the gauge transformations obeyed by a true magnetic vector potential (also sometimes called a “gauge potential” in electrodynamics. If we interpret $\vec{A}_{\text{eff}}$ as a kind of effective vector potential, then $\vec{B}_{\text{eff}}$ will be the “magnetic field” associated with that vector potential.

This analogy gives birth to a variation on the theme of electromagnetism — namely, the above-mentioned Berry magnetism. This “magnetic field” can be thought of as the field due to a (fictitious) magnetic monopole sitting at the center of the earth (whose surface is assumed to be a sphere). Next we show how all this is related to Maxwell’s equations.

B. Accommodating Monopoles into Maxwell’s Equations – Dirac Strings

A seeming contradiction arises here, due to the basic theorem of vector calculus that says that the divergence of a curl is always zero. In this case, $\vec{B}_{\text{eff}}$ is defined as the curl of the vector potential $\vec{A}_{\text{eff}}$ (symbolically, $\vec{B}_{\text{eff}} = \nabla \times \vec{A}_{\text{eff}}$), which would imply that the divergence of $\vec{B}_{\text{eff}}$ (that is, $\nabla \cdot \vec{B}_{\text{eff}}$), must equal zero everywhere, but this is equivalent to saying that there are no point sources of magnetism, ergo no monopoles. How, then, can we have $\vec{B}_{\text{eff}} = \nabla \times \vec{A}_{\text{eff}}$, while at the same time having $\nabla \cdot \vec{B}_{\text{eff}}$ not equal to zero?

The resolution of this contradiction comes from realizing that the vector potential induced by anholonomy — that is, $\vec{A}_{\text{eff}}$ — must be a *singular* function. In other words, nature can reconcile the existence of monopoles with Maxwell’s equations as long as there are vector potentials that “act badly” in certain regions of space.

Fortunately, physicists are already familiar with this type of scenario, thanks to Dirac, who showed, in his 1931 article, that the presence of a monopole amounts to having a vector potential that is singular along a line (usually called a *Dirac string*). Dirac proposed that a magnetic monopole could be envisioned as a semi-infinitely-long thin string of magnetic flux (that is, a line that has one end in a finite region of space, but in the other direction goes out forever — like a water hose with a nozzle that one can hold in one’s hand, but whose source of water is infinitely far away). The accessible end of the Dirac string, where the magnetic flux spills out, acts like a magnetic point charge (in other words, an isolated North or South pole, with no partner anywhere).

Dirac's ideas, building on pioneering work done in 1918 by the German theoretical physicist Hermann Weyl, led him to a very striking result: namely, that if even one magnetic monopole exists, then all electric charge in the universe must be quantized (that is, the electric charge of any subatomic particle must always be an integral multiple of a fixed, fundamental, minimal amount of charge). Put otherwise, the existence of magnetic monopoles would immediately explain why an electron cannot be sliced in half. Dirac was so thrilled with this revelation that he concluded his famous paper with the characteristically British understatement, "One would be surprised if Nature made no use of it."

Paul Dirac forged a completely new style of doing theoretical physics, one that has been deeply influential ever since. By a long-standing tradition, distinguished visitors to the University of Moscow are invited to write on a blackboard a short statement for posterity. The blackboard quote that Dirac left behind after his visit there eloquently sums up his philosophy — namely, "A physical law must possess mathematical beauty."

The elegance, inherent beauty, and universality of anholonomic phenomena in classical and quantum physics, including Yang–Mills theory, which revolutionized particle physics in the 1950s and 1960s, constitute an impressive testimony to what Dirac envisioned and believed in. And yet, we still do not really know why Dirac's philosophy seems to apply so accurately to nature. These elusive ideas are closely related to what Eugene Wigner (a Hungarian-American theoretical physicist who received the Nobel Prize in Physics in 1963) called "the unreasonable effectiveness of mathematics in the physical sciences."

C. Calculating Dirac Strings for Particle in a Magnetic Field

$$\Psi = e^{i\alpha} \begin{pmatrix} (e^{-i\phi/2} \sin \frac{\theta}{2} \\ e^{+i\phi/2} \cos \frac{\theta}{2} \end{pmatrix} \tag{8}$$

One can easily verify that any gauge choice, the corresponding vector potentials exhibit Dirac string singularity, as displayed below. For instance, using spherical polar coordinates $B_x = B \sin\theta \cos\phi$, $B_y = B \sin\theta \sin\phi$ and $B_z = B \cos\theta$, one can show:

$$\Psi(\alpha = \phi/2) \;=\; \begin{pmatrix} \cos\frac{\theta}{2} \\ e^{i\phi}\sin\frac{\theta}{2} \end{pmatrix}; \quad \vec{A}(\alpha = \phi/2) \;=\; -\frac{1}{2B}\frac{1}{(B_z + B)}(-B_y, B_x, 0) \tag{9}$$

This wave function is single-valued at all points on the sphere, but it is ill-defined at $\theta = \pi$, and this is reflected in the corresponding vector potential as a singularity along the line $B_z = -B$, which is the predicted Dirac string.

If we flip the sign of α, we get:

$$\Psi(\alpha = -\phi/2) = \begin{pmatrix} e^{-i\phi}\cos\frac{\theta}{2} \\ \sin\frac{\theta}{2} \end{pmatrix}, \quad \vec{A}(\alpha = -\phi/2) = -\frac{1}{2B}\frac{1}{(B_z - B)}(B_y, B_x, 0) \tag{10}$$

This wave function is single-valued at all points on the sphere, but it is ill-defined at $\theta = 0$, and this is reflected as a singularity in the corresponding vector potential along the line $B_z = +B$, which we identify as another Dirac string.

The above two sample calculations give the flavor of why Dirac's singular strings are inevitable.

III. THE ESAB EFFECT AS AN EXAMPLE OF ANHOLONOMY

In an ESAB setup shown in Fig. (2), with a particle of charge q (e.g., an electron), the effective magnetic field — the field of the monopole — happens to be equal to $\vec{B}_{eff} = \frac{q}{\hbar}\vec{B}$, which is exactly zero everywhere except inside the solenoid — the region where the wave function of the electron vanishes. This makes the region of space that is visitable by the electron not simply connected; and this in turn provides an example of the kind of singularity that necessarily characterizes a Dirac string.

To prove the above result

(1) First prove that solution of Schrödiner equation for an particle encircling a solenoid (so that the magnetic field on the particle is zero) is given by,

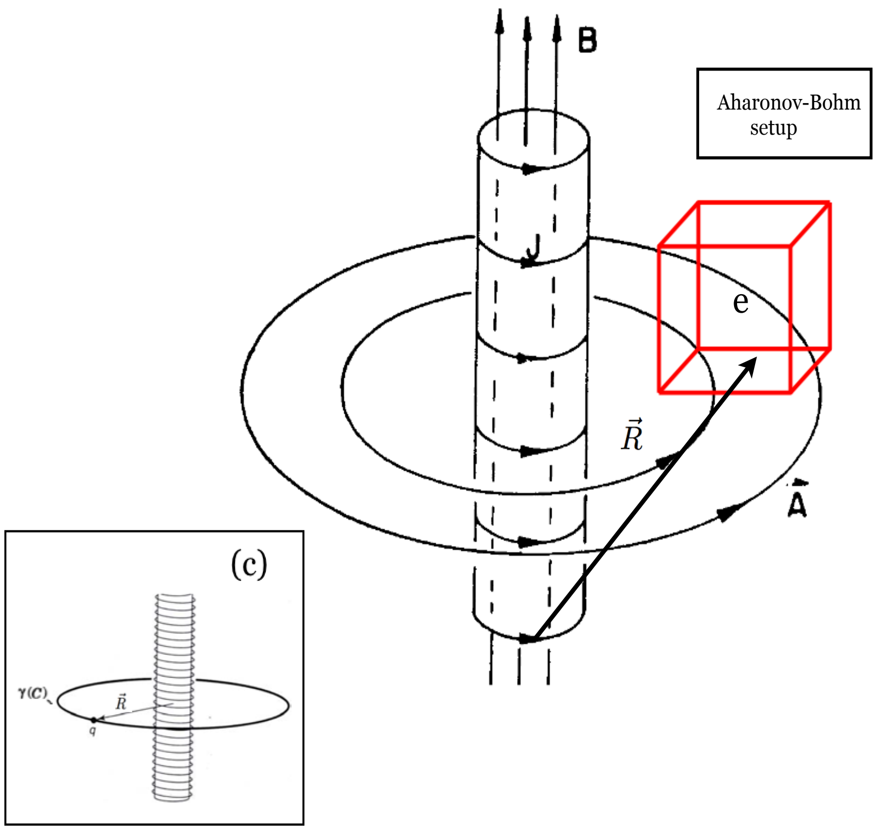

FIG. 2: The curve C denotes the path of the magnetic field. It is very important to distinguish $\vec{B}$ and $\vec{B_{eff}}$.

$$\psi' = e^{i[\frac{q}{\hbar}\oint \vec{A}\cdot d\vec{r}]}\psi, \tag{11}$$

where ψ is the solution in the absence of $\vec{A}$.

(2) Using the set up shown in the figure, where electron is confined in a box, which is centered at point $\vec{R}$ outside the solenoid by a potential $V(\vec{r}-\vec{R})$, we move the box so that $\vec{R}$ will become a function of time, we calculate the Berry phase using the formula (1) derived above.

IV. HOME WORK

Prove Eq. (4), Eq. (10) and (11).