Hofstadter Butterfly and a Hidden Apollonian Gasket

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The Hofstadter butterfly, a quantum fractal made up of integers describing quantum Hall states, is shown to be related to an integral Apollonian gasket with D3 symmetry. This mapping unfolds as the self-similar butterfly landscape is found to describe a close packing of (Ford) circles that represent rational flux values and is characterized in terms of an old (300BC) problem of mutually tangent circles. The topological and flux scaling of the butterfly hierarchy asymptotes to an irrational number $2 + \sqrt{3}$, that also underlies the scaling relation for the Apollonian. This reveals a hidden symmetry of the butterfly as the energy and the magnetic flux intervals shrink to zero. With a periodic drive, the fine structure of the butterfly is shown to be amplified making states with large topological invariants accessible experimentally.

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Hofstadter butterfly[1] is a fascinating two-dimensional spectral landscape, a quantum fractal where energy gaps encode topological quantum numbers associated with the Hall conductivity[2]. This intricate mix of order and complexity is due to frustration, induced by two competing periodicities as electrons in a crystalline lattice are subjected to a magnetic field. The allowed energies of the electrons are discontinuous as electrons in a crystalline lattice are subjected to a magnetic field. The allowed energies of the electrons are discontinuous function of the magnetic flux penetrating the unit cell, while the gaps, the forbidden energies are continuous except at discrete points. The smoothness of spectral gaps in this quantum fractal may be traced to topology which makes spectral properties stable with respect to small fluctuations in the magnetic flux when Fermi energy resides in the gap. The graphic continues to arouse a great deal of excitement since its discovery, and there are various recent attempts to capture this iconic spectrum in laboratory[3]. There is also a renewed interest in related systems exploring competing periodicities[4–6]

Fractal properties of the butterfly spectrum have been the subject of various theoretical studies[7–10]. However, detailed description quantifying self-similar universal properties of the butterfly and the universal fixed point butterfly function has not been reported previously. In this paper, we address the question of spectral and topological self-similarity of the butterfly near half-filling. In contrast to earlier studies for a fixed value of the magnetic flux such as the golden-mean, and thus focusing on certain isolated local parts of the spectrum, we discuss dominant self-similar characteristics that describes the spectrum globally. Our approach here is partly geometrical and partially numerical. Capitalizing on geometrical representation of fractions using Ford circles[11], we obtain the exact scaling associated with the magnetic flux interval as well topological quantum numbers. The nested set of butterflies are described by Descartes’s theorem[12] rooted in 300BC Apollonian problem of four mutually tangent circles, reducing topological landscape to a landscape of a set of close packing of circles. The saga of discrimination of even and odd denomination fractions, both by the number theory and by the quantum mechanics, lies at the heart of the butterfly nesting. Central quantitative outcome is the emergence of a “universal” topological scaling that links the butterfly landscape with dominant scaling characterized by 12-fold quasicrystals to an integral Apollonian gasket[13] with $D_3$ symmetry.

Model system we study here consists of (spinless) fermions in a square lattice. Each site is labeled by a vector $r = n\hat{x} + m\hat{y}$, where $n$, $m$ are integers, $\hat{x}$ ($\hat{y}$) is the unit vector in the $x$ ($y$) direction, and $a$ is the lattice spacing. The tight binding Hamiltonian has the form

$$H = -J_x \sum_r |r + \hat{x}⟩⟨r| - J_y \sum_r |r + \hat{y}⟩⟨r| + h.c. \quad (1)$$

Here, $|r⟩$ is the Wannier state localized at site $r$. $J_x$ ($J_y$) is the nearest neighbor hopping along the $x$ ($y$) direction. With a uniform magnetic field $B$ along the $z$ direction, the flux per plaquette, in units of the flux quantum $\Phi_0$, is $\phi = -\frac{B a^2}{\Phi_0}$. In the Landau gauge, the vector potential $A_x = 0$ and $A_y = -\phi x$, the Hamiltonian is cyclic in $\phi$ so the eigenstates of the system can be written as $\psi_{n,m} = e^{i k_m} \psi_n$ where $\psi_n$ satisfies the Harper equation[1]

$$e^{i k_x} \psi_{n+1}^r + e^{-i k_x} \psi_{n-1}^r + 2 \cos(2\pi n \phi + k_y) \psi_n^r = E \psi_n^r \quad (2)$$

where we have set $J_x = J_y$. Here $n$ ($m$) is the site index along the $x$ ($y$) direction, and $\psi_{n+q}^r = \psi_n^r$, $r = 1, 2, \ldots q$ are linearly independent solutions. In this gauge the magnetic Brillouin zone is $-\pi/q a \leq k_x \leq \pi/q a$ and $-\pi \leq k_y \leq \pi$.

At flux $\phi = p/q$, the energy spectrum has in $q$ bands and $q – 1$ gaps, as seen in Fig. (1). For Fermi level inside each energy gap, the system is in an integer quantum Hall state characterized by its Chern number $\sigma$, the quantum number associated with the transverse conductivity $C_{xy} = \sigma e^2/h$[2]. In fact there are two quantum numbers, $\sigma$ and $\tau$ that are solutions of a Diophantine equation[14], written in two equivalent forms as:

$$r = \sigma p + \tau q, \quad \phi = \sigma \phi_0 + \tau, \quad (3)$$

where $\rho$ is the electron density. For a fixed $p$, $q$ $r$ or $\rho$, the Eq. (3) has a unique solution ($\sigma_0$, $\tau_0$) modulo $q$ and in general there are infinity of solutions ($\sigma$, $\tau$) = ($\sigma_0 – nq$, $\tau_0 + np$), $n = 0, 1, 2, \ldots$. The quantum numbers $\sigma$ that determines the quantized Hall conductivity corresponds to the change in density of states when the magnetic flux quanta in the system is increased by one and whereas the quantum number $\tau$ is the change in density of states when the period of the potential is changed so that there is one more unit cell in the system[8].
A square lattice described by the Hamiltonian (1), supports only \( n = 0 \) solution for the quantum numbers \( \sigma \) and \( \tau \). However, as shown later, finite \( n \) solutions reside in close vicinity of the \( n = 0 \) solution in the butterfly landscape.

Nesting property of the butterfly graph is found to be embedded in a Farey tree organization of various rational magnetic flux values. As discovered by Ford[11], every fraction \( \frac{p}{q} \) can be pictorially represented by a circle, called a Ford circle, of radius \( \frac{1}{2q^2} \), tangent to the horizontal axis, as shown in Fig. (1). A remarkable feature of the Ford circles is the fact that for two different fractions the Ford circles do not intersect and can at most be tangent where the tangencies exist for circles representing rationals which are Farey neighbors[15]. As discussed below, these “tangencies”, described by the (300BC) Apollonian problem will emerge as the central theme in characterizing the butterfly nesting including its topological properties.

One of the key aspects of the nested set of butterflies is the fact that there exist butterflies centered at \( \phi = 0, \phi = \frac{p}{q} = \frac{2n+1}{2m}, (n, m = 1, 2, \ldots), \) that is for all odd values of \( p \) and for all even values of \( q \). These butterflies are microcosm of the butterfly landscape, existing in a magnetic flux window \( \Delta \phi \approx O(q^{-2}) \), as shown below. In other words, there is a butterfly at every scale that is a replica of the butterfly graph. The most dominant quantum Hall states, encoded in each one of such butterfly, has Chern number equal to \( \pm \frac{q}{2} \). The boundaries of each of such a butterfly are located at flux values with odd-denominators, uniquely determined by the center of the butterfly. With \( f_x = \frac{p_x}{q_x}, (x = L, R, c), \) labeling the coordinates of the left and right boundaries and the center, these three are related by the Farey sum, denoted as \( \oplus \):

\[
P_{c} = \frac{PL}{qL} \bigoplus \frac{PR}{qR} = \frac{PL + PR}{qL + qR}.
\]

It is quite remarkable that the parity of the denominator, plays a central role in shaping the butterfly landscape. Firstly, the Harper equation responds quite differently to the even and the odd-denominator flux values. For \( q \)-even, the two bands “kiss” at \( E = 0 \) as the quantum state is degenerate. This degeneracy is lifted as one moves away from this flux value, resulting in the meeting of four gaps, forming a structure resembling the butterfly. For the odd denominator flux, there is a clustering around the band as shown in Fig. 1, known as the Landau levels. Magically, this quantum response is “in tune” with the number theory that also discriminates even and odd-denominator fractions. For every fraction \( f_c \) with even denominator \( q_c \), there exists a unique pair of odd-denominator fractions \( (f_L, f_R) \) with \( q_L < q_c \) and \( q_R < q_c \), and this pair forms the boundaries of the butterfly[11].

We note that the Eq. (4) implies that the three Ford circles representing the center and its left and right boundaries are mutually tangent[11]. These circles which are also tangent to the horizontal line (circle of infinite radius) form what is known as a Descartes’s configuration of collection of four mutually tangent circles whose curvatures \( \kappa_i, i = 1 \text{–} 4 \), satisfy the Descartes’s theorem[12]:

\[
\kappa_\pm = \kappa_1 + \kappa_2 + \kappa_3 \pm 2\sqrt{\kappa_1 \kappa_2 + \kappa_2 \kappa_3 + \kappa_1 \kappa_3}
\]

We identify \( \kappa_1 = 2q^2_L, \kappa_2 = 2q^2_R \) and \( \kappa_3 = 0 \) (the horizontal axis) and \( \kappa_\pm = 2q^2_c \), reducing the Eq. (5) to \( \sqrt{\kappa_\pm} = \sqrt{\kappa_1} \pm \sqrt{\kappa_2} \). The significance of \( \kappa_- \) will unfold shortly.

Above equation along with the Farey neighbor condition[15] determines the magnetic flux interval of the butterfly as:

\[
\Delta \phi = \left| \frac{PL}{qL} - \frac{P_R}{qL} \right| = \frac{1}{qLqR}; \quad \Delta \phi \rightarrow 4q_c^{-2}, \quad q_c \rightarrow \infty,
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![Fig. 1. (color on line) (left) Ford circles along with energy bands highlighted (color coded) for simple flux values: 1/2, 1/3, 2/5, 3/8. The boundaries corresponding to flux values 0/1 and 1/1 are shown with half-circles. On the right, we show the pattern formed by the Ford circles labeled with their curvatures and gaps labeled by the Chern numbers.](image)

We now seek the precise rule for finding the sequence of nested butterflies as we zoom into the flux interval \( \Delta \phi \), determining the center, the left boundary and the right boundary of the butterfly between two successive recursions. We introduce an index \( l \), representing \( l^{th} \) generation or level and the coordinates of the butterfly are specified by \( f_x(l) \), where \( f_c(l) = f_L(l) + f_R(l) \). A close inspection of the butterfly graph where we look at equivalent set of butterflies, see Fig. (2), shows that the boundaries and the center of the butterfly between two successive generations are related by...
two corollaries, continuously while \( \sigma \) gaps that form the central butterfly are, \( \phi \) Diophantine equation, key point being that

\[
\lim_{l \to \infty} a_x(l+1) a_x(l) \approx \zeta^2
\]

The irrational number \( \zeta \) has a continued fraction expansion, \( \zeta = [3, 1, 2, 1, 2, 1, 2, ...] \). We will refer this irrational number as diamond mean. Asymptotically, \( \sigma(l) \to \zeta^{-1}, \tau \to \zeta^{-1} \) and the underlying \( \phi \) interval scales as, \( \Delta \phi(l) \to \zeta^{2l} \). For the butterfly fractal, the entire band spectrum is numerically found to scale approximately as, \( \Delta E(l) \approx 9.87^{-l} \). We note that these scalings describe global features of the butterfly landscape as they are characterized a set of fixed point butterflies with centers at \( \phi_{1,2} = [n_1, n_2, ...] \). We will refer this hierarchy as the diamond hierarchy characterized by the Farey path “LRL”, a symbolic path in the Farey tree that connect two successive generations of butterfly. The hierarchical set of topological integers form a quasicrystal that is associated with dodecahedral symmetry[16].

We briefly note that the butterflies corresponding to golden and silver hierarchies that converges at the set of irrationals \( \phi_1 = [n_1, n_2, 1, 1, ...] \) and \( \phi_2 = [n_1, n_2, 2, 2, ...] \) correspond to the Farey paths \( LRLRLR \) and \( LRRL \), flux scaling ratios \( \left( \frac{1 + \sqrt{5}}{2} \right)^{2l} \) and \( (1 + \sqrt{2})^l \) and energy scaling approximately \( (14)^l \) and \( 38 \). This clearly shows that the Diamond hierarchy is the most dominant hierarchy in the butterfly landscape.

There is another important characteristic that singles out diamond hierarchy in the butterfly fractal. It is intimately related to an Apollonian gaskets, fascinating patterns obtained by starting with four mutually tangent circles shown in the Fig. (3). We first note that the most remarkable aspects of the Apollonian gasket is that if the first four circles have integer curvatures, then every other circle in the packing does too. This fascinating result can be seen from by adding the two roots \( \kappa_4(-) \) and \( \kappa_4(+) \) of Eq. (5) which gives, \( \kappa_4 + \kappa_4 = 2(\kappa_1 + \kappa_2 + \kappa_3) \).

We now consider the case where \( \kappa_1 = \kappa \), which corresponds to an Apollonian with \( D_3 \) symmetry. From Eq. (5) we get,

\[
\frac{\kappa_4}{\kappa_-} = \frac{2 + \sqrt{3}}{2 - \sqrt{3}} = (2 + \sqrt{3})^2
\]

The irrational ratio of these two curvatures shows that there are no integral Apollonian gaskets with exact \( D_3 \) symmetry. However, interestingly, as shown in Fig. (3), in some of the integral Apollonian gaskets, the \( D_3 \) symmetry appears asymptotically.
We study butterfly spectrum for a periodically kicked quantum Hall system\cite{18} where $J_0$ is a periodic function of time $t$ with period-$T$, $J_0 = \lambda \sum_n \delta(t/T - n)$. The discrete translation symmetry $H(t) = H(t + T)$ leads to a convenient basis $\{|\phi_k\rangle\}$, defined as the eigenmodes of Floquet operator $U(T), U(T)|\phi_k\rangle = e^{-i\omega_it}|\phi_k\rangle$. For rational flux $\phi = p/q$, $U$ is a $q \times q$ matrix with $q$ quasienergy bands that reduce to the energy bands of the static system as $T \to 0$. Fig. (4) shows an example of gap amplifications of higher Chern states. This may provide a possible pathway to see fractal aspects of Hofstadter butterfly and engineer states with large Chern numbers experimentally. Recent experiments with ultracold atoms\cite{17} in optical lattices offer a promising means to reveal fractal aspects of the butterfly exposing higher Chern states.

The quantum Hall states are the simplest example of topological insulators that provide topological classification of states of matter. It is conceivable that there may emerge a corresponding classification of butterfly-type structures, based on symmetry and topology, as one considers analogous butterfly graphs in other topological insulators. In other words, topological insulators provide a unique platform to explore the effects of competing periodicities, dictated and intertwined with topology and may shed further light on the dodecagonal and the $D_3$ symmetry that emerged at smallest energy scale in the Hofstadter problem. Finally, recent experimental efforts to see the butterfly in a laboratory are partly driven by a belief the study of butterfly offers the possibilities of discovering materials with novel exotic properties that are beyond our present imagination. Therefore, deeper understanding of the butterfly and related landscapes is a forefront problem and there may be hidden treasures, waiting to be discovered.

\begin{figure}[ht]
\centering
\includegraphics[width=0.5\textwidth]{fig4.png}
\caption{Upper and lower haves respectively show the butterfly spectrum for static and driven cases. Driven part of the figure shows quasienergies for parameter values where the system exhibits a phase transitions for $\phi = 1/3$ from Chern-1 into Chern $-2(= 1 - 3)$ state while Chern $(-2, 1)$ states of $\phi = 2/5$ evolve into Chern $3(= 2 + 5), -4(= 1 - 5)$ states.}
\end{figure}

We next address the question of physical relevance of states of higher topological numbers in view of the fact that size of the spectral gaps decreases exponentially with $\sigma$. We show that by periodically driving the system, we can induce quantum phase transitions to topological states with $n > 0$ solutions of Eq. (3) with dominant gaps characterized by higher Chern numbers. We study butterfly spectrum for a periodically kicked quantum Hall system\cite{18} where $J_0$ is a periodic function of time $t$ with period-$T$, $J_0 = \lambda \sum_n \delta(t/T - n)$. The discrete translation symmetry $H(t) = H(t + T)$ leads to a convenient basis $\{|\phi_k\rangle\}$, defined as the eigenmodes of Floquet operator $U(T), U(T)|\phi_k\rangle = e^{-i\omega_it}|\phi_k\rangle$. For rational flux $\phi = p/q$, $U$ is a $q \times q$ matrix with $q$ quasienergy bands that reduce to the energy bands of the static system as $T \to 0$. Fig. (4) shows an example of gap amplifications of higher Chern states. This may provide a possible pathway to see fractal aspects of Hofstadter butterfly and engineer states with large Chern numbers experimentally. Recent experiments with ultracold atoms\cite{17} in optical lattices offer a promising means to reveal fractal aspects of the butterfly exposing higher Chern states.

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[15] Two rationals $p/q$ and $P/Q$ are called Farey neighbors if they satisfy the equation $|Pq - qP| = 1$.