Entanglement and the Mott transition in a rotating bosonic ring lattice

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We use second- and fourth-order correlation functions accessible in time-of-flight images to investigate the effects of rotation on one-dimensional ultracold bosons confined to a ring lattice. There exists a critical rotation frequency at which the ground state of a weakly interacting and integer-filled atomic gas is fragmented into a macroscopic superposition of two states with different circulation. The formation of such a quantum superposition ("cat") state is accompanied by the opening of a gap in the spectrum, and by a sudden rearrangement of the momentum distribution which lowers the threshold of the Mott insulator transition. We show that both the entangled character of the ground state and the enhancement of quantum correlations can be detected in the density-density correlations of the expanding cloud. Our studies demonstrate the usefulness of these correlations for identifying physics in cold atomic systems.

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I. INTRODUCTION

Ultracold gases loaded into optical lattices are becoming one of the most exciting platforms for exploring complex phases of strongly correlated systems. Experimental control over the parameters of the lattice and the strength of the interatomic interactions have led to the realization of fermionization of bosons, i.e., the Tonks-Girardeau (TG) regime [1,2], and a one-dimensional Mott insulator (MI) state [1]. Recent experimental advances [3,4] are opening a new arena for the investigation of persistent currents and various effects that emerge when cold atoms are trapped in ring-shaped optical lattices. For example, in these systems it is now possible to create an artificial magnetic field by rotating the atomic cloud [5–8] and to study exotic phenomena that emerge when interacting particles are subjected to strong gauge fields.

In this paper, we show how second- and fourth-order correlation functions can be used to investigate the effect of rotation on the ground state properties of bosonic atoms confined in a one-dimensional ring lattice for which a specific experimental optical lattice configuration involving the interference of Laguerre-Gauss and plane wave laser beams have been proposed [4]. Such correlation functions are experimentally accessible in time-of-flight expansion images. When the lattice is rotated with angular frequency $\Omega$, the Coriolis force generates an effective vector potential, which can induce a net circulation or nonzero vorticity. The physics of the interacting quantum many-body system differs considerably between the commensurate case (CO: where the number of atoms, $N$, is an integral multiple of the number of lattice sites, $L$) and the incommensurate case (ICO). The interacting system is characterized by a dimensionless parameter, $\gamma$, which is the ratio of interaction energy to kinetic energy. For the weakly interacting case, $\gamma \ll 1$, recent studies [9,10] have shown that at a critical frequency of rotation, $\Omega_c$, the ground state of CO lattices becomes a macroscopic superposition of two states of opposite circulation (a “cat” state). In this paper we focus on the role of interactions, and map out the dependence on $\gamma$ of the many-body properties of the ground state.

Our most relevant finding is that, for $\gamma \ll 1$, the entangled nature of the ground state of CO systems at $\Omega = \Omega_c$ induces a discontinuous rearrangement of the topology of the momentum distribution. This is accompanied by a substantial increase in the intensity of the shot noise measured in the density-density correlations of the absorption image after expansion [11–15]. ICO systems, on the other hand, are relatively insensitive to rotation, and no abrupt changes are observed in the shot noise intensity.

Beyond the weakly interacting regime, interactions degrade the cat state by pushing atoms away from the two macroscopically occupied quasimomenta. The role of quantum fluctuations on the fragmented ground state is enhanced, the effective quantum depletion is larger, and thus the system is driven to a Mott insulator phase at lower critical value, $\gamma_c$, than the nonrotating ring system. We show that the shift of the Mott insulator threshold can be determined from the variation with $\gamma$ of the density-density autocorrelation peak intensity.

The paper is organized as follows. In Sec. II, we present the basic formalism that describes cold bosonic atoms in a rotating ring lattice. In Sec. III, we study the respective crossings and avoiding crossings of ICO and CO energy levels, and their role on the formation of the cat state. In Sec. IV, we discuss the use of two- and four-point correlations accessible in time-of-flight images to characterize the nature of the many-body ground state. In Sec. V we propose experimental configurations suitable for observing these effects. Section VI presents a summary of our conclusions.

II. BOSONS IN A ROTATING RING LATTICE

We consider a system of $N$ ultracold bosons with mass $M$ confined in a uniform one-dimensional (1D) ring lattice of $L$
sites, with lattice constant \( a \). The ring is rotated in its plane (about the z axis) with angular velocity \( \Omega \). In the rotating frame of the ring, the many-body Hamiltonian is given by

\[
\hat{H} = \int d\mathbf{x} \hat{\Phi}^\dagger \left( -\frac{\hbar^2}{2M} \nabla^2 + V(\mathbf{x}) \right) \hat{\Phi} + \frac{2\pi\hbar^2 \alpha}{M} \hat{\Phi}^\dagger \hat{\Phi} - \Omega \hat{L}_z \hat{\Phi},
\]

where \( \alpha \) is the s-wave scattering length, \( V(\mathbf{x}) \) is the lattice potential, and \( \hat{L}_z \) is the angular momentum. Assuming that the lattice is deep enough to restrict tunneling to nearest-neighbor sites, and that the band gap is larger than the rotational energy, the bosonic field operator, \( \hat{\Phi} \) can be expanded in Wannier orbitals confined to the first band, \( \hat{\Phi} = \sum_j \hat{\phi}_j W_j(\mathbf{x}) \), with \( W_j(\mathbf{x}) = \exp\left( -iM \int x' A(x') \cdot d\mathbf{x'} \right) W_j(\mathbf{x}) \). Here \( W_j(\mathbf{x}) \) are the Wannier orbitals of the stationary lattice centered at the site \( j \), with angular velocity \( \Omega \), and \( \hat{\phi}_j \) is the bosonic annihilation operator of a particle at site \( j \). In terms of these quantities, the many-body Hamiltonian can be written, up to onsite diagonal terms which we neglect for simplicity, as [8,16]

\[
\hat{H} = -J \sum_j e^{-i\theta \hat{a}_j^\dagger \hat{a}_{j+1} + e^{i\theta \hat{a}_{j+1}^\dagger \hat{a}_j} + U/2} \sum_j \hat{n}_j \hat{n}_j - 1. \tag{2}
\]

Here \( \hat{n}_j = \hat{a}_j^\dagger \hat{a}_j \), \( \theta \) is the effective phase twist induced by the gauge field, \( \theta = M/\hbar \int \nabla \cdot \mathbf{A}(x') \cdot d\mathbf{x'} = \frac{\mu_0 \epsilon \epsilon_0}{h} \). \( J \) is the hopping energy between nearest-neighbor sites: \( J = \int d\mathbf{x} W_j^\dagger \left( -\frac{\hbar^2}{2M} \nabla^2 + V(\mathbf{x}) \right) W_{j+1} \), and \( U \) the on-site interaction energy, \( U = \frac{4\mu_0 \epsilon \epsilon_0}{h} \int d\mathbf{x} W_j^4 \).

### III. Ground State Degeneracy and Formation of a Cat State

The many-body properties of a rotating bosonic ring at the critical frequency for vortex formation, \( \Omega = \Omega_c \), differ considerably depending upon the CO or ICO character of the system. The differences can be related to the existence of an avoided crossing in the CO case and a true crossing in the corresponding ICO system. In this section we study this point in both weak and strong interaction limits, where analytic solutions are available.

To understand the effect of rotation on the ground state properties of atoms in a rotating ring lattice, it is convenient to write the many-body Hamiltonian in terms of quasi-momentum operators defined as

\[
\hat{b}_q = \frac{1}{\sqrt{L}} \sum_{j=1}^L \hat{a}_j e^{-i(2\pi qj/L)}, \tag{3}
\]

where \( 2\pi q/(dL) \) is the discrete quasi-momentum (in \( h \) units) and \( q \) is an integer, \( q=0, \ldots, L-1 \). In the quasi-momentum basis \( \hat{H} \) is given by

\[
\hat{H} = -2J \hat{H}_J + \frac{U}{2L} \hat{H}_s = \sum_{q=0}^{L-1} \frac{E_q}{2L} \hat{b}_q^\dagger \hat{b}_q + \frac{U}{2L} \sum_{q=0}^{L-1} \sum_{s=0}^{L-1} \hat{b}_q^\dagger \hat{b}_s \hat{b}_s^\dagger \hat{b}_q \mid q+s \rightarrow q \rangle \langle q \rangle.
\]

Here \( E_q = -2J \cos(2\pi q/L - \theta) \) are single-particle energies and the notation \( \| L \rangle \) indicates modulo \( L \). The modulus is taken because in collision processes the quasimomentum is conserved up to an integer multiple of the reciprocal lattice vector \( 2\pi/dL \) (the Umklapp process).

We begin by considering the noninteracting limit, \( U = 0 \). While in a static ring, \( \theta = 0 \), the ground state corresponds to a macroscopically occupied state with well-defined quasimomentum: \( q_0 = m \) and \( q_0 = m+1 \) for \( \Omega < \Omega_c \) and \( \Omega > \Omega_c \), respectively. However, the fact that exactly at the critical frequency, \( \Omega = \Omega_c \), the single particle energies \( E_{qm} \) and \( E_{qm+1} \) become degenerate leads to a \((N+1)\)-fold degenerate many-body ground state. The \( N+1 \) degenerate energy levels correspond to the states with \( n \) atoms in \( q = m \) and \( N-n \) atoms in \( q = m+1 \), with \( n = 0, \ldots, N \). We denote such states as \( |n, N-n\rangle = \frac{1}{\sqrt{n!(N-n)!}} \hat{b}_m^\dagger \hat{b}_{m+1}^\dagger |0\rangle \).

In the presence of interactions, one expects the degeneracy to be lifted. For small \( \gamma = U/J \), one can use lowest order perturbation theory to account for the interaction effects. In view of the absence of direct coupling between the different, \( |n, N-n\rangle \) states, i.e., \( \frac{U}{2L} \langle n', N-n'| \hat{H} |n, N-n\rangle = \frac{U}{2L} \langle N(N-1) - 2n(n-N) \rangle \), the energy shifts are simply given by:

\[
E_{n}^{(1)} = NE_n + \frac{U}{2L} \langle N(N-1) - 2n(n-N) \rangle. \tag{7}
\]

Therefore, at lowest order in perturbation theory, the states with minimal energy become those corresponding to \( n = 0 \) and \( n = N \). These two states remain degenerate and higher order perturbation theory is needed to break the degeneracy. At this point one must distinguish between CO and ICO cases.

In the ICO case, it has been shown in Ref. [9] that there is no coupling between \( |N, 0\rangle \) and \( |0, N\rangle \) and therefore, these states remain degenerate for all values of \( U \). In the CO sys-
The absence of coupling in the ICO case can be intuitively understood by considering the total quasimomentum (in ħ units), \( K = 2\pi/(Ld) (\sum_{n} \hat{n}_n \hat{\beta}_n^\dagger + \sum_{n} \hat{n}_n \hat{\beta}_n^\dagger) \). The many-body Hamiltonian given by Eq. (4) exhibits a block diagonal form if the quasimomentum Fock states are ordered according to \( K \) [17]. In the CO case, \( N = \tilde{n}L \) with \( \tilde{n} \) an integer and thus the states \(|N,0\rangle \) and \(|0,N\rangle\), have total quasimomentum \( K = 2\pi/(Ld) \tilde{n} \rangle \rangle L = 0 \) and \( K = 2\pi/(Ld) (m+1) \rangle \rangle L = 0 \), respectively, and thus both of them belong to the \( K = 0 \) block. On the other hand, in the ICO case, \( N = \tilde{n}L + \Delta N \), and hence the two states \(|N,0\rangle \) and \(|0,N\rangle\), belong to different blocks as \( \langle \tilde{n}L + m\Delta N \rangle \rangle L \neq \langle (m+1) \rangle \rangle L + (m+1) \rangle \rangle L \). Thus these two states are not coupled by interaction.

An effective Hamiltonian can be constructed by coarse-graining the effect of other states on \(|N,0\rangle \) and \(|0,N\rangle\), with the resulting form

\[
H_{\text{eff}} = \begin{pmatrix} E^{(1)}_N & V_{12} \\ V_{12}^* & E^{(1)}_N \end{pmatrix},
\]

where

\[
V_{12} = \Delta E = \frac{U}{2L} \langle \tilde{n}(L-1) \rangle \begin{pmatrix} A \end{pmatrix},
\]

and \( A = \sum_{i,j} \langle \hat{n}_i \hat{n}_j \rangle \hat{H}_{ij} \). In this expression, the \( \hat{H}_{ij} \) transition matrix elements introduced by the interaction term of the Hamiltonian, and \( \hat{n}_i \) are either \( E^{(1)}_n \) or noninteracting many-body eigenenergies depending upon whether the intermediate states are or are not in the \(| n, N \rangle \rangle L \) manifold. The factor \( \tilde{n}(L-1) \rangle \rangle L \) corresponds to the minimum number of collision processes necessary to connect the states \(| N,0\rangle \) and \(|0,N\rangle\) and the sum is over all the different paths that generate such couplings. Here \( \tilde{n} = N/L \).

For CO systems, the nonzero value of \( V_{12} \) couples the two macroscopically occupied states, leading to the opening of an energy gap, \( \Delta E \), and to the formation of a cat state,

\[
|\Psi\rangle = \frac{a_1 |0,N\rangle + a_2 |N,0\rangle}{\sqrt{2}}
\]

with \( \frac{a_1}{a_2} = \frac{V_{12}}{E^{(1)}_N} \) (see [9]). The opening of the energy gap is key to the attainment of a stable cat state. Figure 1 shows the variation of \( \Delta E \) with interaction for \( \tilde{n} = 1 \) and \( L = 9 \), based on a numerical calculation. The power-law dependence on \( \gamma \) is in agreement with Eq. (9).

In contrast, in the ICO case, the \(|N,0\rangle \) and \(|0,N\rangle\) states remain degenerate as they are not coupled by interactions. Due to the rotational invariance of the Hamiltonian, the ground state may still be taken to be an equal superposition of the two degenerate states. However, due to the absence of an energy gap, any disturbance or infinitesimal asymmetry always present in real situations will cause the collapse of the cat state.

Up to this point, we have restricted the analysis to the weakly interacting regime. As \( \gamma \) increases, perturbation theory breaks down, and states with quasimomentum different from \( m, m+1 \) populate the ground state. In the intermediate regime, the energy spectrum can no longer be understood in terms of single particle solutions, and it must be determined by numerical calculations. However, in the strongly interacting limit, \( \gamma \gg 1 \), the bosonic gas becomes fermionized, and the Bose-Fermi mapping [18] can be used to quantitatively study the lowest energy levels.

We now discuss the fermionized limit, treating first the case \( N \ll L \). In this limit the ground state of the strongly interacting bosonic system is obtained by filling all the single-particle states up to the Fermi energy. Hence, the total ground state energy is just the sum of the lowest \( N \) single-particle energies. At this point, however, it is important to recall the distinction between cases of even and odd values of \( N \), deriving from the boundary conditions implied by fermionization. For odd \( N \), periodic boundary conditions must be imposed; for even \( N \), antiperiodic boundary conditions are required [18]. These conditions lead to a single particle spectrum of the form \( E_q = -2J \cos(\pi/4(q-\sqrt{q^2-4m^2})) \) for odd \( N \), and of the form \( E_q = -2J \cos(\pi/4(q-\sqrt{q^2-4m^2})) \) for even \( N \). Note that if \( |L| = \pi \) (i.e., \( \Omega \neq \Omega_L \)), the ground state is nondegenerate. However, as shown in Fig. 2, at \( |L| = \pi \) the ground state is always twofold degenerate for \( N < L \): \( E_q = E_{q+1+2m} \) for odd (even) \( N \). In contrast, for the CO case, the band is completely filled and hence the ground state remains unique. Moreover, there is a large energy gap of order \( U \), and the system is a Mott insulator independent of the value of \( \theta \).

\[\text{FIG. 1. Energy gap } [\Delta E, \text{ see Eq. (9)}] \text{ between the ground state and the first excited state in units of } J \text{ for } L=9 \text{ and } N=9 \text{ as a function of } \gamma=U/J. \text{ The solid grey line is proportional to } (\gamma)^{-1}.\]

\[\text{FIG. 2. Single particle fermionic spectrum at } \Omega_L \text{ for even and odd } N, \text{ the crosses indicate occupied states (energies below the Fermi energy). The figure shows that for } N < L \text{ the ground state is twofold degenerate.}\]
For the general case $N > L$, it has been shown in Ref. [19], that if $N$ is written as $N = nL + \Delta N$, with $n$ the largest integer smaller than $N$, the extra $\Delta N$ bosons can be treated as hard core bosons and mapped to ideal fermions on top of a sea of $nL$ frozen atoms. The effect of the other $nL$ atoms on the mobile $\Delta N$ atoms is just to renormalize their effective tunneling energy by $Jn$. Thus, up to a constant energy shift of $Un(N+n\Delta N-L)/2$ and a renormalization of $J$, the spectrum is identical to that described above for $N < L$. The degeneracy exhibited by the ICO system and the uniqueness of the CO system therefore remain in the strongly interacting limit.

**IV. MOMENTUM DISTRIBUTION AND NOISE CORRELATIONS**

In a typical experiment, atoms are released from the lattice potential by turning off the trapping lasers. The atomic cloud then falls as it expands, and is photographed after it enters the ballistic regime. By assuming that the atoms are noninteracting from the time of release, properties of the initial state can be inferred from the spatial images: the density distribution image reflects the initial quasimomentum distribution, $n(Q) = \sum_\rho \rho_\rho$, $Q = 0, 1, \ldots, L-1$ and the density-density correlations, namely the noise correlations, reflect the quasimomentum fluctuations, $\Delta(Q_1, Q_2)$.

$$\Delta(Q_1, Q_2) = \langle \hat{n}(Q_1)\hat{n}(Q_2) \rangle - \langle \hat{n}(Q_1) \rangle \langle \hat{n}(Q_2) \rangle.$$  

In Eq. (11) we have assumed that both, $Q_1, Q_2$, lie inside the first Brillouin zone.

Figure 3 shows the quasimomentum distribution as a function of $\gamma$ for CO and ICO systems. In these plots we shift the origin by $m$, recalling $\theta = \frac{2\pi}{\Gamma} + \frac{2\pi \Delta}{L}$. In addition, at $\Omega_c$ we superimpose an infinitesimal symmetry breaking field that randomly shifts $\Omega_c \rightarrow \Omega'_c$ or $\Omega_c \rightarrow \Omega''_c$ to mimic real experimental situations. In agreement with our analysis, in the weakly interacting regime the quasimomentum distribution is peaked at $Q = q_0$, the condensate mode, with $q_0 = m$ for $\Delta < \pi$ and at $q_0 = m + 1$ for $\Delta > \pi$. At $\Delta = \pi$, $n(Q)$ looks different for the ICO and CO cases. While the ICO system always exhibits a sharp interference peak (sometimes peaked at $Q = m$, others at $Q = m + 1$) the CO case exhibits a broader and flatter profile (except at $U = 0$) as the total population is equally distributed between the $Q = m$ and $Q = m + 1$ components.

To contrast this feature further, in Fig. 4 we display the fractional number of atoms in the $Q = q_0$ mode, and $Q = q_0 \pm 1$ as a function of $\Delta \theta$ and $\gamma$, with $\pm$ determined by $\Delta \theta \equiv \pi$. While the ICO system always exhibits a single macroscopically occupied mode, the CO system exhibits an abrupt redistribution of the population of atoms as $\theta$ approaches $\Omega_c$. This rearrangement changes the topology of the $\hat{n}(Q = q_0, U, \theta)$ and $\hat{n}(Q = q_0 \pm 1, U, \theta)$ surfaces: they touch at the critical rotation frequency. Furthermore, because a gap only opens up for the CO case, as demonstrated in Sec. III, the compressibility, $d\mu/d\Gamma$, which is the variation of the chemical potential with the density $\bar{n} = N/L$, shows a singular behavior at integer fillings (see Fig. 5).

**FIG. 3.** (Color online) Quasimomentum density vs $\gamma$ and $Q$ [the discrete quasimomentum is $h2\pi Q/(LD)$ with $Q$ an integer and $d$ the lattice spacing]: upper row CO system, lower row ICO system. See Eq. (5) for the definition of $m$.

**FIG. 4.** (Color online) Quasimomentum density vs $\gamma$ and $\Delta \theta$. The top (black) curve is for the quasimomentum density evaluated at $Q = q_0$ and the bottom (red) evaluated at $Q = q_0 \pm 1$ ( $\pm$ depending if $\Delta \theta \equiv \pi$). $2\pi q_0/L$ is the wave vector (in lattice units) associated with the condensate component. Here $L = 9, N = 9$ (left-hand side), $N = 8$ (right-hand side).

**FIG. 5.** (Color online) Variation in the chemical potential, $\mu$, vs density, $\bar{n}$, and $\gamma$ at $\Omega_c$. For this calculation we used $L = 6$. 

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The topological changes in the quasimomentum distribution, the opening of a gap in the spectrum and the nonanalyticity in the compressibility are reminiscent of the Lifshitz transition in the ideal fermionic system. In the Lifshitz transition a change in the pressure beyond a critical value results in an abrupt change in the topology of the Fermi surface, which in turn leads to a nonanalytic behavior of the density of states, and a nonanalytic compressibility. A typical example of the Lifshitz transition is the disruption of a neck of the Fermi surface. From this perspective, rotation mimics the effect of pressure and induces a Lifshitz-like transition in bosonic atoms confined to a ring-shaped geometry. However, in contrast to the Lifshitz transition, which takes place in noninteracting systems, in the present case the change of the topology is induced by interactions. In this respect, the topological changes are more reminiscent of the topological phase transitions reported in Ref. [21] where electron-electron interactions, rather than the band structure, are responsible for fermions migrating to other regions of reciprocal space.

In practice however, if instead of a single image, one takes the average over many realizations, the information in the momentum distribution about the topological rearrangement is lost, since on average both the weakly interacting ICO and CO systems show the same quasimomentum profile. In contrast, the topological changes and fragmented nature of the CO state have distinctive signatures in the noise correlation pattern.

Figure 6 shows the quantum noise interferometry pattern observed as we vary the rotation frequency and the interaction strength. Away from criticality, one sees the emergence of bright interference fringes as the Mott insulator transition is approached. Furthermore, at criticality, a high-contrast pattern of both positive and negative fringes is seen for rather small values of the interaction, heralding the formation of a highly entangled state. By comparing the interference pattern between the CO and ICO cases, we see that the latter remains almost unaffected by rotation except for an overall translation. The CO interferogram at $\Omega_c$ is in agreement with Eqs. (10), which yields
\[ \Delta(m,m) = \Delta(m+1,m+1) = -\Delta(m,m+1) = \frac{N^2}{4} \] (12)

and zero otherwise. It has reflection symmetry along the \( Q_z = -Q_1 + 2m+1 \) axis, demonstrating equal population of the \( Q = m, m+1 \) quasimomentum components. For \( \Omega \approx \Omega_c \), on the contrary, there is preferential population of the \( Q = m \) or the \( Q = m+1 \) components.

The results displayed in Fig. 6 are consistent with the notion that as \( \gamma \) increases, interactions deplete the macroscopically occupied modes. Other modes become populated, and at a critical value \( \gamma_c \), the CO system undergoes a quantum phase transition to a Mott insulator state. Deep in the Mott regime, irrespective of the \( \theta \) value, the ground state evolves into an equal superposition of all the different quasimomentum components. Earlier studies performed for the static system [24] suggest that the onset of the Mott insulator transition is heralded in the noise correlations by the development of a maximum in the autocorrelation, \( \Delta(q_0,q_0) \), as the interaction is tuned. The development of such a maximum is particularly relevant in finite systems, where the condensate population decreases gradually and thus the momentum distribution fails to provide a distinctive signature of the transition.

In Fig. 7 we plot \( \Delta(Q_1,q_0) \) vs \( Q_1 \) and \( \gamma \) for different \( \Delta \theta \). While the noise pattern for the ICO system is rather insensitive to rotation, in the CO system, as \( \Delta \theta \) approaches \( \pi \), not only do the intensities of \( \Delta(q_0,q_0) \) and \( |\Delta(q_0+1,q_0)| \) increase, but also the position of the maximum shifts towards lower \( \gamma \) values, suggesting that the enhancement of correlations induced by rotation lowers the threshold \( \gamma_c \) for the Mott insulator phase.

To quantify the enhancement in Fig. 8 we plot, for a given \( \Delta \theta \), the maximum value of \( \Delta(q_0,q_0) \) as \( \gamma \) is varied and the \( \gamma_c \) value at which the maximum is reached. The shift in the critical value in finite systems is in qualitative agreement with a Gutzwiller ansatz analysis which is based on the assumption that the wave function can be approximated as a product of the wave functions at the various lattice sites. The Gutzwiller ansatz predicts a decrease of the critical value of the Mott insulator transition threshold by a factor of \( \cos(\Delta \theta/L) \) with respect to the static case, \( \gamma_c = \gamma_c^{\text{static}} \cos(\Delta \theta/L) \). The mean field shift, however, is much

![FIG. 7. (Color online) \( \Delta(Q_1,q_0) \) vs \( Q_1 \) and \( \gamma \) for \( L=9 \) and \( \Delta \theta = 0, 0.4, 0.5 \) (left-hand, middle, and right-hand panels, respectively). 2\( \pi/Lq_0 \) is the wave number associated with the condensate in lattice units. The (a)–(c) panels are for the CO system with \( N=9 \). In this case, \( \Delta(q_0,q_0) \) exhibits a maximum at the onset of the MI for \( \Delta \theta = \pi \). At \( \Delta \theta = 0 \) a jump at \( U>0 \) occurs signaling the formation of a cat state. The panels (d) and (e) are for the ICO case with \( N=8 \). On the contrary in this case noise correlations are almost insensitive to rotation.](image)

![FIG. 8. (a) Maximum intensity reached by \( \Delta(q_0,q_0) \) and (b) \( \gamma \) value at which the maximum takes place. Away from criticality, the maximum takes place at the onset of the Mott insulator transition, however at \( \Omega = \Omega_c \), such maximum is overwhelmed by a jump close to \( U=0 \) which signals the formation of a cat state.](image)
smaller than that value found in our numerical calculations but this is expected as mean field theories only provide a qualitative analysis in 1D systems. Appendix A summarizes various details of this analysis.

The sensitivity of noise correlations to the formation of the cat state is also seen in Figs. 7 and 8(b), as the maximum in the autocorrelation function at $\gamma_c$ is overwhelmed at criticality by an abrupt jump in $\Delta(q_0, q_0)$ at $U/J \ll 1$ when the cat state is formed. The fact that noise correlations are sensitive both to the onset of the Mott insulator transition and to the cat state formation—situations where the many-body system is expected to be maximally entangled—suggest that such experimental accessible correlations can be used as a probe of entanglement.

In the strongly correlated regime, physical insight about the noise interference pattern can be gained by using the Bose-Fermi mapping. While the CO system enters the Mott insulator phase and exhibits a noise pattern with positive interference peaks along the $Q_i = \|Q_i\|_2$ axis with amplitudes independent of the $Q_1$ value, the ICO system remains superfluid and the signature of it on the noise pattern is the existence of a dominant peak at $Q_1 = Q_2 = q_0$. As the basic features of the noise interference pattern, except for an overall translation, remain very similar to the static case, we refer the interested reader to Refs. [11,24] for further details.

Numerical calculations of noise correlations, as described above, have been done for a system containing few lattice sites. In view of this, the questions regarding the generality of our conclusions and reliability of the numerical results are important. We partially address this issue by studying the weakly interacting regime, away from criticality and by comparing the numerical results with analytical calculations within the Bogoliubov approximation. As shown in Appendix B, the finite-size calculations are in fact a reasonable approximation at least in the weakly interacting regime away from criticality where the Bogoliubov approximation is valid.

V. EXPERIMENTAL CHALLENGES

We now discuss the feasibility of experimental observation of the cat state formation and the enhancement of correlations close to $\Omega_c$.

Optical ring lattices can be created experimentally by the interference of a Laguerre-Gaussian (LG) laser beam with a plane wave copropagating, along the $z$ direction [3,4]. Moreover by reflecting the combined beam back on itself one can achieve confinement along the $z$ direction, thus creating a stacked array of disk shaped traps. By controlling the tunneling between the disks, and making it much smaller than the corresponding tunneling within each ring, one can implement an array of effective 1D ring lattices. The lattices can be rotated by introducing a phase shift in one of the two beams that compose the interferogram.

Even though the realization of such lattice configurations is certainly reasonable with current technology, the experimental observation of the cat state formation is challenging. Some of the reasons are (a) control over the number of atoms is mandatory as it only occurs for commensurate lattices; (b) precise control over the rotation frequency is required due to the narrow resonance width at $\Omega_c$; (c) an almost perfect realization of the ring geometry is crucial due to the fact that any decoherence effects or asymmetries that induce perturbations larger than the energy gap will collapse the cat state. Note that from Eq. (7) it is clear that the magnitude of the gap rapidly decreases with the number of atoms and sites.

In the following we will discuss possible solutions to overcome such limitations.

(a) The Mott insulator phase transition, in translational invariant systems requires a commensurate number of atoms, as does formation of a cat state. However, the Mott insulator has been experimentally observed without controlling the atom number, thanks to the harmonic confinement potential used in the laboratory to collect the atoms. The reason for this is that in the presence of a confining potential the commensurability criteria is no longer required: by tuning the trapping frequency and lattice depth, it is always possible to realize a Mott insulator and control the atom density at the trap center for an arbitrary number of atoms [19].

It might be possible to form a Mott insulator with exactly $n$ atoms per site as the initial configuration and then use it to load a commensurate number of atoms in the ring lattice. Also, an additional reduction of the fluctuations in the number of atoms might be achievable by combining standard BEC techniques with single-atom counting. The usefulness of the latter was recently demonstrated in Ref. [22] where a reduction in the number fluctuation below the Poissonian limit for average numbers between 300 to 60 atoms was reported.

(b) and (c) Reference [9] investigated the phase sensitivity required for the formation of a cat state in CO systems. This study highlighted the experimental difficulty of creating a cat state as the number of atoms increases. For example, even for just $N=12$ and $L=3$, it was found that in order to have a fidelity above 50%, $\delta \theta = \Delta \theta - \pi$, must be less than $10^{-5}$ at $\gamma = 0.1$. However, it was also noted there that if instead of atoms with short-ranged interactions, cold particles interacting via long-range interaction are used, such as molecules with large dipole-dipole interactions, an energy gap between the states with all atoms having the same quasimomenta and the rest of the Hilbert space opens up, making the formation of the superposition less phase sensitive. Such a gap will also make the system more insensitive to local perturbations induced by any external inhomogeneities in the ring.

In addition, probing entanglement formation via noise correlations requires sufficiently high experimental resolution along with sufficiently many atoms to overcome background and photon noise problems. However, in view of rather rapid experimental developments in the field and recent progress in noise-correlations interferometry [12–15] we are optimistic that experimental control to observe the entanglement formation and enhancement of quantum correlations via noise correlations may be possible in the near future.
We conclude this discussion by emphasizing that even though the shift in the $\gamma_c$ disappears in the thermodynamic limit, our analysis applies to finite-size systems and should be observable in practical experiments.

VI. SUMMARY

In this paper, we have investigated the effect of rotation on the ground state properties of bosonic atoms confined in a one-dimensional ring lattice. In these optical lattice arrangements the physics of interacting many-body systems differs considerably depending on the commensurability of the numbers of atoms and lattice sites. In integer-filled systems, at a critical value of the rotation frequency, weak interatomic interactions open a gap in the excitation spectrum, fragment the ground state into a macroscopic superposition of two states with different circulation and generate a sudden change in the topology of the momentum distribution. We showed that the entangled nature of the ground state induces a strong enhancement of quantum correlations and decreases the threshold for the Mott insulator transition, effects that can be experimentally measured by using noise-correlation interferometry. In contrast, the incommensurate system is rather insensitive to rotation and regardless of the strength of interactions, at $\Omega_c$, it remains gapless with two degenerate ground states. In this case no fragmentation takes place and the noise interference pattern does not show any important changes with respect to the nonrotating system. Note that even though most of the numerical results presented in the paper are for low-density systems, $N \ll L$, we have also tested our conclusions for higher densities.

Further investigation of the similarities between the topological redistribution of the bosonic momentum distribution, and the topological changes in the Fermi surface that occur in the Lifshitz transition in ideal fermionic systems is needed. In both cases the abrupt change in topology is accompanied by the opening of a gap and a nonanalytic behavior of the transition. However, in contrast to the latter, the former is induced by atomic interactions. Our studies raise important open questions regarding the onset and nature of the Mott insulator transition at $\Omega_c$, since we have a transition from a gapped, fragmented and incompressible state to a Mott insulator, rather than the standard transition from a superfluid state. We hope that our suggestion of a Lifshitz-like transition in bosonic systems will stimulate a new line of research in the condensed matter community.

Finally, we have also discussed some of the practical challenges that will be confronted by the experimental attempts to observe the phenomena described in this paper.

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APPENDIX A: THE GUTZWILLER ANSATZ AND THE MOTT TRANSITION

Recalled that CO systems enter a Mott insulating phase at a critical value of the interaction. We now explore the effect of rotation on the Mott transition by using the mean-field Gutzwiller ansatz [23] which is based on the assumption that wave function can be approximated as a product of the wave functions at the various lattice sites,

$$|\psi_G(t)\rangle = \prod_{i=0}^{N} \left( \sum_{n=0}^{\infty} f_n^{(i)} |n_i\rangle \right).$$

(A1)

Here $|n_i\rangle$ are Fock states with $n$ atoms at site $i$. Deep in the Mott phase, the only relevant states are those characterized by $\bar{n}$, $\bar{n}+1$, and $\bar{n}-1$ atoms per site. Due to the translational invariance of the ring and periodic boundary conditions we expect plane waves to yield the solution with minimal energy. Assuming $f_n^{(i)} = \sqrt{\frac{n}{N}} e^{i\epsilon_n}$, $f_{\bar{n}+1}^{(i)} = \sqrt{\frac{\bar{n}+1}{N}} e^{i\epsilon_{\bar{n}+1}}$, and $f_{\bar{n}}^{(i)} = \sqrt{\frac{\bar{n}}{N}} e^{i\epsilon_{\bar{n}}}$ with $\alpha = 2\pi \bar{n}/L$, $\epsilon \equiv 0$, and $f_n^{(i)} = 0$, and minimizing the energy $E = \langle \psi_G | H | \psi_G \rangle$ with respect to $\epsilon$ and $\alpha$ (as it becomes $\beta$ independent), we find that the probability of having sites with $\bar{n} \pm 1$ atoms vanishes at the critical value $\gamma_c$ given by

$$\gamma_c = 2 \cos(\Delta \theta/L)(\sqrt{n} + \sqrt{n+1})^2,$$

(A2)

where $\alpha = 2 \pi n/L$ for $0 \leq \Delta \theta < \pi$ and $\alpha = 2 \pi (n+1)/L$ for $\pi < \Delta \theta < 2 \pi$. The Gutzwiller ansatz predicts a decrease of the critical value of the Mott insulator transition threshold by a factor of $\cos(\Delta \theta/L)$. Note that in the thermodynamic $L \rightarrow \infty$ the predicted shift disappears. The mean field shift is much smaller than the value found in our numerical calculations. However, this is expected, since, first, mean field theories only provide a qualitative analysis in 1D systems and second, exactly at $\Omega_c$, the model predicts a degenerate ground state, contradicting our previous analysis and suggesting the break down of the Gutzwiller ansatz at the critical rotation.

APPENDIX B: THE BOGOLIUBOV APPROXIMATION

We use the Bogoliubov approximation to calculate noise correlations. This provides further information about the properties of quantum noise interferometry, particularly those pertaining to the weakly interacting regime.

In the Bogoliubov approximation the field operator is approximated by a $c$-number plus small fluctuations, $\hat{a}_n = (z_n + \hat{\delta}_n) e^{-i \mu t}$, with $\mu$ the chemical potential of the system. Replacing this ansatz in the Bose-Hubbard Hamiltonian, Eq. (2), and keeping terms up to second order in $\delta$ yields a quadratic Hamiltonian which can be diagonalized by the canonical transformation $\hat{\delta}_n = \sum_{\omega} \sqrt{\alpha_{\omega}} \hat{a}_\omega e^{-i(\omega-n)\mu t} - \sqrt{\alpha_{\omega}} \hat{a}_\omega^\dagger e^{i(\omega-n)\mu t}$. Here $\alpha_{\omega}$ are the so-called quasiparticle energies and $(\hat{u}_n^\dagger, \hat{v}_n^\dagger)$ the quasiparticle amplitudes. The latter are constrained to preserve the Bose commutation relations of the quasiparticle operators, $\hat{u}_n^\dagger \hat{u}_m^\dagger - \hat{v}_n^\dagger \hat{v}_m^\dagger = \delta_{nm}$ and $\sum_{nm} \hat{u}_n^\dagger \hat{v}_m^\dagger = \delta_{nm}$. We expect that the Bogoliubov analysis can provide a fair description of the many-body physics
in the weakly interacting regime, when there is a unique macroscopically occupied mode, i.e., away from $\Omega_c$. The plane wave ansatz $z_n = e^{i(2\pi/L)nq_0}\sqrt{n_0}$, $u_n^a = \frac{1}{L} e^{i(2\pi/L)nq_0}\sqrt{\epsilon}\eta_n$, and $v_n^a = \frac{1}{L} e^{i(2\pi/L)nq_0}\sqrt{\epsilon}\eta_n$, with $q_0$ and $q$ integers, satisfies periodic boundary conditions and diagonalizes the quadratic Hamiltonian if

$$\mu = -2J \cos \left[ \frac{\Delta \theta}{L} \right] + n_0 U, \quad (B1)$$

$$\omega_q = \Lambda_q + \sqrt{\epsilon^2_q + 2U\eta}\epsilon_q, \quad (B2)$$

$$v_q^2 = v_q^2 - 1 = \frac{\epsilon_q + U\eta}{2\sqrt{\epsilon^2_q + 2U\eta}\epsilon_q} - \frac{1}{2}, \quad (B3)$$

$$n_0 = \bar{n} - \frac{1}{L} \sum_{q \neq 0} v_q^2. \quad (B4)$$

Here $\Lambda_q = 2J \sin \left( \frac{\Delta \theta}{L} \right) \sin \left( \frac{\pi}{L} \right)$, $\epsilon_q = 4J \sin \left( \frac{\pi}{L} \right) \cos \left( \frac{\Delta \theta}{L} \right)$, $\bar{n} = N/L$ is the total density and $n_0$ is the condensate density. To obtain the minimal energy as usual we have chosen $q_0 = m$ if $0 \leq \Delta \theta < \pi$ and $q_0 = m + 1$ if $\pi < \Delta \theta < 2\pi$.

We note that Eq. (B2) corresponds to the excitation spectrum of a current carrying condensate where $\frac{\Delta \theta}{L}$ is the wave vector associated with the current [26,27]. If such wave vector (in lattice units) exceeds a critical value, $\pi/2$, which is equal to one-quarter of the reciprocal lattice constant, then the coherent motion of the condensate becomes unstable, resulting in the loss of superfluidity. Such a dynamical instability was observed experimentally [28] by measuring loss of coherence as a function of the condensate momentum. It is of classical origin, in the sense that it is seen in the Gross-Pitaevskii (GP) equations of motion and differs from the Landau dissipation mechanism, occurring when the velocity of the condensate is larger than the sound speed [27]. As the system enters the strongly interacting regime, detailed theoretical [29] and experimental studies [30] have shown that quantum fluctuations smoothen the sharp classical transition and lead to current decay at smaller critical wave vector than predicted using the classical GP equations alone. This behavior is interesting in the point of view of the rotating ring lattice as $\frac{\Delta \theta}{L}$ can only exceed $\pi/2$ for small systems with $L \leq 3$ (remember $0 \leq \Delta \theta < 2\pi$), however, if the critical velocity is decreased by interaction, the ring might exhibit dynamical instabilities even for larger values of $L$. This point may warrant further investigation.

From Eqs. (B1)–(B4), one can calculate analytic expressions for the momentum distribution and the noise correlations in terms of quasiparticle energies and amplitudes,

$$\hat{n}(Q) = n_0 \delta_{q_0}Q + v_{q_0}^2 Q, \quad (B5)$$

$$\Delta(q_0, q_0) = 2 \sum_{q \neq 0} u_{q_0}^2 u_{q_0}^2, \quad (B6)$$

FIG. 9. (Color online) Comparisons between the exact numerical solution (color lines) and the Bogoliubov approximation (black lines). The latter are only shown for moderated values of $\gamma$ as for large $\gamma$ the Bogoliubov approximation breaks down. Top: momentum distribution vs $\gamma$. We use no symbols for $n(q_0)$, triangles for the smallest distribution between $n(m)$ and $n(m + 1)$, and boxes for the sum over the population of other quasimomenta. Bottom: noise correlation vs $\gamma$. We use no symbols for $\Delta(q_0, q_0)$, triangles for $\Delta(m, m + 1)$ and boxes for the smallest distribution between $\Delta(m + 1, m + 1)$ and $\Delta(m, m)$.

$$\Delta(q_0, Q) = -2 u_{q_0}^2 u_{q_0}^2 \delta_{q_0}Q Q, \quad (B7)$$

$$\Delta(Q, Q) = u_{q_0}^2 u_{q_0}^2 Q^2 \delta_{q_0}Q Q + \delta_{q_0}Q Q, \quad (B8)$$

where $u_{q_0}^2 u_{q_0}^2 = \frac{\gamma^2 q^2}{4(\omega_q - \Lambda_q)^2}$ and $Q \neq q_0$. The Bogoliubov analysis predicts a sharp interference peak in the momentum distribution at $q_0$, the condensate quasimomentum. As the ratio $\gamma$ increases, its intensity decreases and its width increases due to quantum depletion in agreement with Fig. 3. On the other hand, noise correlations are directly related to the quantity $u_q^2 u_q^2$, the so-called anomalous average [25]. It is related to the the many-body scattering matrix which accounts for the modification of binary collision properties due to the presence of other surrounding atoms. The only nonvanishing correlations are $\Delta(q_0, q_0)$, $\Delta(q_0, Q)$, $\Delta(Q, Q)$, and $\Delta(2q_0 - Q, Q)$ with $Q \neq q_0$. The Bogoliubov analysis probes pair excitations of the condensate and has a negative intensity. The anticorrelation can be understood as a consequence of the destructive interference between the excited pairs generated when two condensate atoms collide. $\Delta(Q, Q) = \Delta(2q_0 - Q, Q)$ probe collisions between atoms out of the condensate; these processes are less frequent than condensate collisions and that is why they have one-half of the intensity of $\Delta(q_0, Q)$. $\Delta(q_0, q_0) = 2 \sum_{q \neq 0} u_{q_0}^2 u_{q_0}^2$ has the larger intensity as its value is 2 times the sum of the intensities of the other $\Delta(Q, Q)$ peaks. Figure 9 shows a comparison between analytic results obtained within the Bogoliubov approximation and the exact numerical diagonalization of Eq. (2) for a finite ICO system. The agreement in the small $\gamma$ region for a system of only nine lattice sites is suggestive evidence that our analysis based on a finite lattice with few atoms captures relevant features.
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