Chern and Majorana Modes of Quasicrystals

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The topology of quasicrystals is found to have a novel manifestation in the spatial profile of band edge states as topological invariants transform peaks into doublets of size equals the Chern number. The Chern-dressed peaks form a self-similar pattern encoding topological fingerprints at all length scales. For quasicrystals exhibiting localized states, fluctuations about exponentially localized zero modes describe the onset to topological transition where Majorana modes delocalize. These exotic modes can be captured in their entirety using $U(1)$ symmetry breaking perturbation that supports both the Chern and the Majorana modes. Here topological transition is accompanied by localization as edge-localized modes move to the interior, loosing topological protection.

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The revelation that quasicrystals belong to topologically nontrivial phases[1] of matter is an exciting new development that opens new avenues in the frontiers of topological insulators. Quasicrystals (QC) are fascinating ordered structures exhibiting self-similar properties and long range order with regular Bragg diffraction.[2] Characterized by a reciprocal space of dimensionality higher than the real space, these systems do not exhibit crystallographic rotational symmetries. Topological insulators are exotic states of matter that are insulators in the bulk but conduct along the edges[3], characterized by topologically protected gapless boundary modes. These edge modes are a manifestation of the nontrivial band structure topology of the bulk[4] and their number equals[5] the topological integer, the Chern number. We will refer to these modes as edge-Chern modes.

Key to the topological characterization of Quasicrystals is the translational invariance that shifts the origin of quasiperiodic order[1] and manifests as an additional degree of freedom relating QCs to higher dimensional periodic systems. These shifts of origin known as phasons produce QCs that look locally different but belong to the same isomorphism class[6]. Topological description of QCs require an ensemble of such systems and can be characterized by Chern numbers in view of their mapping to higher dimensions. Explicit demonstration of transport, mediated by the edge mode was shown in a beautiful experiment by pumping light across a photonic QC.[1] Harper and Fibonacci models, the two iconic examples of QCs, are shown[7, 8] to be topologically equivalent and this equivalence is preserved irrespective of whether the quasiperiodic disorder appears in the diagonal or in the off-diagonal term.

This paper elucidates a novel manifestation of topology that is unique to QCs. In 1D QCs, we show that the band edge modes encode topological invariants in their spatial profiles. Characterized by goldenmean incommensurability, the central peaks and the sub-peaks separated by Fibonacci distances of the band edge states split into doublets of size equals the Chern number. These Chern-dressed dimerized peaks accompanying edge-Chern modes are found in Harper, Fibonacci model as well as in a generalized model that interpolates between the two. In other words, in QCs, the topology introduces a new length (equals the Chern) that together with competing incommensurate periodicities of the QC provides a hierarchical manifestation of topological invariants.

QCs also provide a new perspective on Majorana modes, zero-energy topologically protected modes at the ends of an infinitely long system with open boundary condition that have been the subject of very intense studies due to possible applications to quantum computing.[10] In QCs where quasiperiodicity induces localization, we show that the fluctuations about exponentially localized zero energy modes describe onset to a topological phase transition where edge localized move to the interior of the chain. To study Majorana away from criticality, we consider a perturbed Harper model with broken $U(1)$ symmetry system. The system describes a $p$-wave superconducting quantum wire and also anisotropic XY spin-$1/2$ model in a spatially inhomogeneous quasiperiodic magnetic field. These QCs host both the Majorana and Chern modes, two very different type of topologically protected edge modes. Topological phase transition in these quasiperiodic system is accompanied by a localization transition where the zero energy modes move from the edge to the interior of the chain. Edge-Chern modes remain unperturbed by the localization and the band-edge modes localize at two-sites separated by a distance equals the Chern number. Although the system supports both the Majorana and Chern modes, the gap hosting the Majorana forbids Chern to exist in the same gap.

We consider a 1D chain of spinless fermionic atoms in a lattice described by the Hamiltonian,

$$H(\phi) = -\sum_{s} c_{s}^\dagger c_{s+1} + \text{h.c.} - \sum_{s} V_{n}(\phi)c_{s}^\dagger c_{s}$$  (1)

Here, $c_{s}^\dagger$ is the creation operator for a fermion at site $s$. In our studies we have investigated a generalized potential that interpolates between the Harper and the Fibonacci model[7], however, for our presentation here we will restrict to the Harper model with $V_{n} = 2\lambda \cos(2\pi(\sigma n + \phi))$, incommensurate po-
tential characterized by an irrational number \( \sigma \) which we take to be the golden mean \((\sqrt{5} - 1)/2\). Here \( \lambda \) controls the strength of quasiperiodic disorder and \( \phi \) is an arbitrary phase. The Eigenvalue equation, namely the Harper equation,

\[
\psi_{n+1}^{r} + \psi_{n-1}^{r} + 2\lambda \cos(2\pi(n + \phi))\psi_{n}^{r} = E\psi_{n}^{r} \tag{2}
\]

exhibits a self-similar spectrum and wave functions at \( \lambda = \pm 1 \)[11, 12]. This self-dual point is the critical point for quasiperiod disorder-induced quantum phase transition from extended ( \( \lambda < 1 \) ) to exponentially localized phase ( \( \lambda > 1 \))[11] with localization length \( \xi = \ln(\lambda) \).

The incommensurate system is studied by approximating the golden mean, \( \sigma \) by a sequence of rational approximates, the ratio of two consecutive Fibonacci numbers: the Fibonacci sequence is defined by \( F_0 = 0, F_1 = F_2 = 1, \) and \( F_n = F_{n-1} + F_{n-2} \). For any rational approximant \( \sigma = F_{n-1}/F_n \), the system consists of \( F_n \) bands and \( F_{n-1} \) gaps.

In parallel with the well known[4] formalism of quantized Hall conductivity for the 2D bulk problem, one can define an adiabatic conductivity \( \sigma_{\phi} = e^2/h \psi^{l} \) for a 1D ensemble \( H(\phi) \) of chains with periodic boundary condition,

\[
\sigma_{\phi} = \frac{e^2}{h} \text{Im} \sum_{l=1}^{r} \int d\phi \int dk \sum_{n=1}^{q} \partial_{\phi}(\psi_{n}^{l})^{*} \partial_{\phi}\psi_{n}^{l}. \tag{3}
\]

where \( k \) is the Bloch index, \( r \) labels the gap characterized by Chern number \( c_r \) and integration over \( \phi \) corresponds to an ensemble average of set of chains related to each other by translation using phason shifts [6] controlled by \( \phi \). However, the quantity in the square bracket, the Chern density[1] in Eq. (3) is independent of \( \phi \), and therefore Chern number can be associated with any \( H(\phi) \). This is in contrast with the rational \( \sigma \), where the Chern density and the energy \( E \) depend upon \( \phi \) and therefore Chern number is associated only with the whole family of 1D systems, and that a single periodic system belongs to the trivial phase.

For a finite chain, gaps host edge-Chern modes that traverse the gaps as \( \phi \) varies. The topological invariants manifest as the number of adiabatic excursions across the QC as the phase varies in its entire range. It can be shown that the Cherns form a pseudorandom pattern, as the gap index \( r \) varies, given by \( c_r = F_{n}[\frac{1}{2} - < r\sigma + \frac{1}{2} >] \), an exact solution of the Diophantine equation.[9] Here \(< x > \) means fractional part of \( x \).

We now show that topology introduces a new length, equals the Chern in QCs exhibiting power-law or exponential localization. This manifestation of topology exists in bulk states that appear at the band-edges, namely the states that borders the gaps and hence reside (energetically) in the immediate vicinity of the tail of the edge-Chern modes. We will refer these modes as band-Chern modes. Figures 1 display the spatial profiles of topologically trivial (ground state) and the band-Chern modes with \( c_r = 1, 2, 3 \). As further illustrated in Figs (2) and (3), major as well as the secondary peaks exhibit Chern-dressing, where the symmetry in the doublet structure, seen clearly in the main peak, is only restored asymptotically.

**FIG. 1**: (color online) With \( \lambda = 1 \), (a) describes spatial profile of topologically trivial ground state while the figures (b), (c) and (d) respectively illustrate “Chern dressing” of the band-Chern modes with \( c_r = 1, 2, 3 \). Here the central as well as secondary peaks are transformed into double-peak structure of size equals the Chern. In contrast to the central peak, doublet structure is in general asymmetrical and this asymmetry is restored as one moves far from the main peak as seen in Fig (2).

**FIG. 2**: (color online) Figure shows the self-similar wave functions for the topologically trivial state (top) and band-Chern state with Chern number 4 (bottom) at \( \lambda = 1 \). Topology splits the peaks into doublets and the dimerized pattern preserves self-similarity.
FIG. 3: (color online) Blowup of the central peak for ground (blue) and band-Chern (red) for $\lambda = 1$ (top). The black double arrow-line shows the unusual peak separation of 9 in Fibonacci landscape. The bottom shows the corresponding plots for $\lambda = 1.1$ (red) and $\lambda = 10$ (black) illustrating topological encoding irrespective of the amount of disorder.

We recall that all previous scaling analysis of the wave functions for QCs have been carried out for special points of the spectrum such as mid-band points[12] or the band edge points corresponding to maximum or minimum energy states. Such states are topologically trivial. These studies describe incommensurate states as consisting of a central or main peak and a sequence of sub peaks at Fibonacci distance from the central peak as shown in Fig. 2. The subpeaks have intensity that approach a well defined universal ratio (See Fig. (2) describing the ratio of the subpeaks to the central peak, for peaks far from the central peak. The wave functions display self-similarity as the structure around subpeaks approach a scaled version of the structure around the central peak. It is quite intriguing that a new length in the Fibonacci landscape preserves the self-similar fractal pattern that characterizes quasiperiodic systems as structures around subpeaks approach a scaled version (with a universal scaling ratio) of the peaks around the central peak.

Fig. 3 compares the spatial profile near the main peak for topologically trivial and non-trivial band-Chern state with Chern number four. One of the interesting outcomes of the self-similar fractal pattern with Chern-dressed Fibonacci peaks is the existence of certain non-Fibonacci peak separations such as 9 that are unrelated to the Chersns. Common myth that peak separations in goldenmean QCs must be a Fibonacci is not valid for topological states. Such anomalous peaks follow the period-3 pattern[12] intertwined with the self-similar pattern. Although simple consideration that pattern inbetween two Fibonacci should be symmetrical about the center accounts for such unusual peak separations, in our detailed study, only few such peak separations are observed and hence remain mysterious.

As illustrated in Fig. 3, the self-similar spatial profiles decay exponentially for $\lambda > 1$. For infinite disorder, these states evolve into a dimer of size equals the Chern number, referred as Chern-dimer mapping.[15] This mapping was demonstrated explicitly in the large $\lambda$ limit as the system maps to a two-level system.[15] For $\lambda \rightarrow \infty$, the mapping is also valid for rational $\sigma$. However, for the incommensurate case, the mapping persists for finite $\lambda > 1$ and transforms into global mapping at $\lambda = 1$ existing at all length scales. In other words, it is the localization (power-law or exponential) that is key to the fact that topological invariants manifests as a new length in the QCs. Chern dimer mapping along with self-similar structure of the ground state suggests that spatial profile of the band-Chern mode can be thought of as a convolution of a minimum energy non-topological state and the $\lambda \rightarrow \infty$ state. The ansatz $\Psi_n^* (\lambda = 1) = \sum_m \psi_m^*(\lambda \rightarrow \infty) \psi_{n-m}(E_{min})$ provides a good approximation to the exact solution for a range of energies.[16]

We will now discuss QCs that in addition to Chern modes
also support Majorana modes.[10] We first note that in QCs such as the Harper model exhibiting exponential localization, fluctuations about exponentially decaying envelope[13] of zero energy state describes Majorana at the onset to their extinction. From Eq. (2), it follows that the fluctuations $\eta_n$ about exponentially decaying envelope, $\psi_n = e^{-n/\xi} \eta_n$ satisfy the following equation,

$$e^{-\xi} \eta_{n+1} + e^{\xi} \eta_{n-1} + 2\cos(2\pi(\sigma + \phi)) \eta_n = E \eta_n \quad (4)$$

where the localization length $\xi = ln \lambda$. For $E = 0$, this equation describes a fermionic representation of the zero energy state of a spin-1/2 anisotropic XY-chain with exchange interactions $J_x = e^{-\xi}$ and $J_y = e^\xi$ along the $x$ and the $y$ axis respectively, in a spatially modulated transverse magnetic field $V_n$.[14] The system is described by the Hamiltonian,

$$H = \frac{1}{2} \sum \left( J_x f_{n+1}^\dagger c_n + J_y f_{n-1}^\dagger c_n + V_n c_n^\dagger c_n \right) \quad (5)$$

The system also describes a $p$-wave superconducting quantum wire with superconducting gap parameter $\Delta = J_y - J_x$, the spin-space anisotropy. The eigenvalue equation for the system is a coupled set of equations,

$$J_x f_{n+1} + J_y f_{n-1} + V_n f_n = E g_n \quad (6)$$

$$J_y g_{n+1} + J_x g_{n-1} + V_n g_n = E f_n \quad (7)$$

Here $(f_n, g_n)$ represents two-component wave function of the superconducting chain with particle-hole symmetry. For $\Delta \neq 0$, this perturbed Harper with broken $U(1)$ symmetry ($c_j \rightarrow e^{i\eta} c_j$) has been shown to exhibit localization transition at a critical value of $\lambda = \lambda_c$, where $\lambda_c = J_y$, beyond which all states are exponentially localized.[14] This transition is accompanied by closing of the gap at $E = 0$. In a finite chain, the gap at $E = 0$ hosts Majorana fermions, localized edge modes existing for $\lambda < \lambda_c$. Comparing Eqs. (4) and (5), we note that the mapping between zero energy state to that of Harper is valid precisely at critical point of the topological transition. Therefore, the critical point, $\lambda = \lambda_c$ of the system describing onset to localization also describes the critical point of the topological transition that is captured both in Harper system and in perturbed Harper with broken $U(1)$ symmetry. To study topological phase we study the anisotropic model (Eq. (7)), confining mostly to small $\Delta$ values which leaves the Chern modes of the Harper almost unchanged and in addition supports the topological edge mode at $E = 0$.

Below criticality ($\lambda < \lambda_c$), the anisotropic spin-chain is gapped at $E = 0$, making it topologically distinct from Harper equation. The system supports Chern modes in all gaps except the central gap that hosts Majorana. Figure (4) summarizes the energy spectrum and the wave functions of the Chern and the Majorana modes. The zero-energy edge localized modes become extended (critical) at $\lambda_c$ and localize in the interior of the chain for $\lambda > \lambda_c$. This may provide a rather unique system to trace Majorana modes as it leaves its fingerprints in the nontopological phase. In contrast, the edge-Chern modes remains immune to quasiperiodic disorder. The band-Chern modes exhibit topological fingerprints at the critical point (as peaks split into a double-peak) and also in the localized phase where the state localizes at two sites at a distance $c_x$ apart.

Interplay between topology and quasiperiodicity provides a remarkable example of competing periodicities where Cherm, the aliens adapt to self-similar environment controlled by the Fibonacci. The topology dominates the local length scales, maintaining global Fibonacci landscape. The mathematical framework to describe renormalization studies of QC states with non-trivial topology is an open problem that will be of interest to wide audience interested in fractal characteristics of incommensurate systems.

Quasicrystals described by the Harper equation have been realized in ultracold atomic gases[17]. Superconducting quantum wires have been shown to be promising candidates for realizing Majorana modes.[18] Quasicrystalline superconducting chains or anisotropic spin chains that effectively describe two-dimensional periodic systems[7] offer a fascinating new set of possibilities to study topological states supporting both the Majorana and the Cherm. The Majorana exists in the central gap, the only gap in this multiband system that does not support the Chern modes. The fact that the Majorana and the Chern modes do not coexist (in the same gap) is an intriguing result and its deeper implications remains to be understood.

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References: