Metal-Insulator Transition Revisited for Cold Atoms in Non-Abelian Gauge Potentials

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We discuss the possibility of realizing metal-insulator transitions with ultracold atoms in two-dimensional optical lattices in the presence of artificial gauge potentials. For Abelian gauges, such transitions occur when the magnetic flux penetrating the lattice plaquette is an irrational multiple of the magnetic flux quantum. Here we present the first study of these transitions for non-Abelian $U(2)$ gauge fields. In contrast to the Abelian case, the spectrum and localization transition in the non-Abelian case is strongly influenced by atomic momenta. In addition to determining the localization boundary, the momentum fragments the spectrum. Other key characteristics of the non-Abelian case include the absence of localization for certain states and satellite fringes around the Bragg peaks in the momentum distribution and an interesting possibility that the transition can be tuned by the atomic momenta.

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Ultracold atom research presents many avenues to explore problems at the forefront of theoretical physics. One recent line of investigation is the generation of artificial “magnetic” fields in an electrically neutral quantum gas [1–3], which have the same effect on neutral atoms as a magnetic field has on charged particles. For atoms confined in an optical lattice, these synthetic gauge fields can be realized by imposing appropriate complex phase factors upon the amplitudes for tunneling of atoms between neighboring sites. This has renewed interest in the substantial theoretical literature of diamagnetism of electrons in incommensurate two-dimensional lattices [4], a system which exhibits rich phase structure but has seemed difficult to realize with any clarity in a condensed-matter system. Atoms in optical lattices may now provide a vehicle for experimental exploration of this subject.

Furthermore, it also seems possible to create non-Abelian gauge fields in optical lattice systems [5], fields in which the components of the vector potential are non-commuting operators rather than scalar functions. Such fields are of fundamental importance in condensed-matter and particle physics, and ultracold atomic physics now offers the prospect of controlled, well-characterized experiments on them [6]: for example, the observation of analogue “magnetic monopoles” [7] and non-Abelian interferometry [5]. Recently, it was suggested that the artificial magnetic fields may provide a laboratory realization of the fractal energy spectrum of an electron gas in a two-dimensional lattice, the Hofstadter butterfly [8], and the “Hofstadter moth,” the corresponding spectrum for the $U(2)$ gauge fields [1,5]. Another interesting feature that can be realized in these systems is the metal-insulator transition when the magnetic flux $\alpha$ per lattice plaquette is an irrational multiple of the magnetic flux quantum [4]. In contrast to the Mott transition, which has its origin in particle interactions [9], this transition is induced by competing periodicities associated with the cyclotron frequency and the frequency of motion of a particle in a periodic lattice [8].

The localization transition obtained by tuning the ratio of the tunneling along the two directions of the lattice has been extensively studied [4]. Here, we revisit the transition for the case when the “magnetic field” corresponds to $U(2)$ gauge fields. We describe various novelties associated with the transition in the non-Abelian case and discuss the experimental feasibility of seeing these transitions in two-dimensional optical lattices.

Our starting point is a tight binding model (TBM) of a particle moving on a two-dimensional rectangular lattice $(x, y)$, with lattice constants $(a, b)$, and nearest-neighbor hopping characterized by the tunneling amplitudes $J, J \Lambda$. When a weak external vector potential, $\tilde{A}(x, y) = (A_x, A_y, 0)$, is applied to the system, the Hamiltonian is given by

$$H = -J \left[ \cos \left( \left( p_x - \frac{e}{c} A_x \right) \frac{a}{\hbar} \right) + \Lambda \cos \left( \left( p_y - \frac{e}{c} A_y \right) \frac{b}{\hbar} \right) \right],$$

where $\tilde{p}$ is the momentum operator. For simplicity, we work in the Landau gauge: $\tilde{A}(x, y) = (A_x, A_y, 0)$, where $A_x$ is a constant and $A_y$ depends only on the $x$ coordinate. The resulting Hamiltonian is cyclic in $y$ and thus the eigenfunctions are proportional to plane waves in the $y$ direction. Denoting the transverse wave number of the plane wave as $\tilde{k}_y = k_y / b$, the wave function can be written as follows: $\Psi(ma, nb) = e^{i\tilde{k}_y \cdot n} g_m$, with $x = ma$ and $y = nb$. For the Abelian gauge case, with $\tilde{A} = (0, Bx, 0)$, one obtains the Harper equation [4]

$$g_{m+1} + g_{m-1} + 2\Lambda \cos(2\pi m \alpha - k_y) g_m = E g_m,$$

where $\alpha = Bb / (\hbar c / e)$ and $E(\alpha)$ is the energy in units of $-J$. The energy spectrum is the union over $k_y$ of individual
energy spectra of the Harper equation. For rational $\alpha = p/q$, the spectrum consists of $q$ bands which are usually separated by gaps. As $k_y$ varies, the bands shift and their length may change, but they do not overlap, except at the band edges. For irrational $\alpha$, the spectrum is independent of $k_y$ and the system exhibits a metal-insulator transition at $\Lambda = 1$. As we discuss below, this insensitivity of the spectrum and the localization transition to the transverse wave number is lost when the gauge potential is non-Abelian.

We obtain a non-Abelian vector potential for ultracold atoms in quite a natural way, when we treat systems in which the atoms have two degenerate internal states. The most general vector potential will couple the two states, which the atoms in quite a natural way, when we treat systems in non-Abelian. The atoms in quite a natural way, when we treat systems in Abelian.

The wave function $\Psi(m, n) = e^{i\beta_m} g_m$ is a two-component vector $g_m = (\theta_m \eta_m)$ that satisfies the following equations:

$$
\begin{align*}
\left( \frac{\theta_{m+1}}{\eta_{m+1}} \right) + \left( \frac{\theta_{m-1}}{\eta_{m-1}} \right) - \left( \frac{\epsilon_m^a \eta_m}{\epsilon_m^a \theta_m} \right) &= 0, \quad (2)
\end{align*}
$$

where $\epsilon_m^a = E - 2\Lambda \cos(2\pi \alpha_m m - k_y)$.

These coupled equations can be decoupled resulting in a pair of TBMs:

$$
\begin{align*}
The m_{1+1} \theta_{m+2} + \epsilon_m^a \theta_{m+2} &= C_m(\alpha_1, \alpha_2) \theta_m, \quad (3a) \\
\epsilon_m^a \eta_{m+1} + \epsilon_m^a \eta_{m+2} &= C_m(\alpha_2, \alpha_1) \eta_m, \quad (3b)
\end{align*}
$$

where $C_m(\alpha_1, \alpha_2) = \epsilon_m^a \epsilon_{m+1}^a \epsilon_{m+2}^a - \epsilon_m^a \epsilon_{m+1}^a - \epsilon_{m+1}^a$.

For $\alpha_1 = \alpha_2$, the non-Abelian problem reduces to the Abelian problem described by the Harper equation [Eq. (1)]. Calculation of the spectral properties and the detailed exploration of their localization transition in the non-Abelian problem is done using various methods which include a transfer matrix approach [5,8], mapping to a dynamical system [10], and the scaling arguments from Fibonacci renormalization theory [11]. In our study we fix $\alpha_1$ to be the golden mean, $\alpha_1 = (\sqrt{5} - 1)/2$. The non-Abelian problem is studied for various values of $\alpha_2$, and we describe our results for $\alpha_2 = \frac{1}{2}, \frac{1}{4}, \text{and } \frac{1}{4}$. In the transfer matrix analysis of the uncoupled Eqs. (3a) and (3b), we consider a sequence of periodic systems obtained by rational approximants $\alpha_i = p_i/q_i$, which corresponds to truncating the continued fractional expansion of $\alpha_1$ (and also of $\alpha_2$ for the irrational case). For the golden mean, rational approximants are the ratio of two successive Fibonacci numbers $F_n$: $F_1 = 1, F_2 = 1, F_{n+1} = F_n + F_{n-1}$. Allowed energies correspond to unit modulus eigenvalues of the product of transfer matrices of period $Q$, the least common multiple of $q_1$ and $q_2$. In general, Eqs. (3a) and (3b) give two sets of allowed energies depending upon whether $m$ is even or odd, and therefore the full spectrum of the non-Abelian system described by Eq. (2) is obtained by the intersection of the two spectra obtained for the even- and odd-site $\theta$ (or $\eta$) equation. The localization transition that exists provided at least one of the $\alpha_i$ is irrational can be realized in the numerical simulation by studying a sequence of periodic systems as we approach the irrational limit.

Localization properties, particularly the phase diagram of the system, can be best studied by mapping the TBMs [Eq. (3a)] to a dynamical system, a discrete driven map obtained by defining $x_m = \theta_{m-1}$. The $\theta$ equation becomes $x_{m+2} = -1/[R_m x_m - f_m]$. Here $R_m = \epsilon_m^{a+1}/\epsilon_m^{a-1}$ and $f_m = C_m(\alpha_1, \alpha_2)/\epsilon_m^{a-1}$. This mapping of a TBM to a dynamical map is extremely useful in determining the localization transition as the absolute value of the Lyapunov exponent $\gamma$ is related to the inverse localization length $\xi$ of the TBM, $\xi^{-1} = \gamma/2$. It should be noted that the maps are nonchaotic, and hence the Lyapunov exponents are always negative.

In contrast to the Abelian case, the spectrum and the localization properties of states in the non-Abelian case are strongly dependent on the transverse momentum $k_y$.

Figure 1 shows the fragmented energy spectrum of the $\alpha_2 = \frac{1}{2}$ case as the transverse wave vector is varied. The $k_y/2\pi = \frac{1}{4}$ emerges as a special wave number which characterizes the ground state and is also important for the central state ($E = 0$) as three different bands touch at $E = 0$. Furthermore, the minimum energy develops a

FIG. 1 (color online). Energy spectrum viewed as a function of $k_y$ for $\alpha_1 = \frac{1}{144}$, $\alpha_2 = \frac{1}{4}$, and $\Lambda = 0.3$. Light dots show the corresponding Abelian case for $\alpha_1 = \alpha_2 = 89/144$ where the spectrum does not depend upon $k_y$. Minimum energy shown by a solid line shows the step structure with $\frac{1}{8\pi} = \frac{1}{2}$ being the global minimum or the ground state.
step structure and the number of steps increases with \( \Lambda \) (Fig. 2), and this structure resembles a fractal set at the onset to localization. The origin of these discontinuities is twofold: first, only a fraction of the full range of \( k_y \) results in the allowed energies, and second, the incompatibility of the two sets of energy solutions corresponding to the even- and the odd-site equations. However, there exists a characteristic value of the wave number for which even-odd–site equations result in identical set of allowed energies. For \( \alpha_2 = \frac{1}{2} \), this selected set of wave numbers is \( \frac{k_y}{2\pi} = \frac{1}{7} \) (mod 1). For \( \alpha_2 = \frac{1}{2^3} \), which exhibits analogous step structure, the characteristic wave numbers are given by \( \frac{k_y}{2\pi} = \frac{1}{8} \) (mod \( \frac{1}{4} \)).

Figure 2 shows the localization aspect of the spectrum. In stark contrast to the Harper equation where all states localize simultaneously at \( \Lambda = 1 \), localization in the non-Abelian case begins at the band edges at \( \Lambda = 0.48 \) for \( \alpha_2 = \frac{1}{2} \). In this case, all states except those in the central band localize before or at \( \Lambda = 1 \). As \( \Lambda \) increases, the width of the central band decreases, and as \( \Lambda \to \infty \), only \( E(k_y = \pi/2) = 0 \) remains extended, a fact that can be inferred from Eq. (2) since it corresponds to \( \epsilon_m^\alpha = 0 \). This defiance of localization of the central band is a unique feature of the \( \alpha_2 = \frac{1}{2} \) case. For \( \alpha_2 = \frac{1}{2^3} \), localization begins at \( \Lambda = 0.24 \) and all states localize before \( \Lambda \leq 0.83 \).

To understand the origin of the selected set of wave numbers, we now focus on the TBM corresponding to the \( E = 0 \) state, which is always an eigenvalue of Eq. (2) for irrational \( \alpha_1 \). For \( \alpha_2 = \frac{1}{2} \), the \( \theta \) equation for \( E = 0 \) reduces to

\[
\theta_{m+2} + \theta_{m-2} + 2\Lambda_{\text{eff}} \cos(2\pi \alpha_1 m - k_y) \theta_m = (-2)\theta_m, \tag{4}
\]

where \( \Lambda_{\text{eff}} = 2\Lambda^2 \cos(k_y) \). Thus, the \( E = 0 \) state of the non-Abelian system is mapped to the \( E = -2 \) state of a Harper-like Eq. (4) where the strength of the on-site quasiperiodic potential is a sinusoidal function of \( k_y \). Diagonalization of Eq. (4) (using \( \alpha_1 = 89/144, 144/233, \ldots \), and periodic boundary conditions) for various values of \( \Lambda_{\text{eff}} \) shows that \( E = -2 \) is an eigenvalue of the system provided \( \Lambda_{\text{eff}} \leq 0.48 \). Thus the boundary curve \( E(k_y, \Lambda) = 0 \) for the delocalized phase of the \( E = 0 \) state is given by \( 2\Lambda^2 \cos(k_y) = 0.48 \). This result is in perfect agreement with the numerically obtained boundary curve in Fig. 3 using dynamical mapping and renormalization methods.

Very direct evidence of the importance of the transverse wave number in the localization transition emerges for \( \alpha_2 = \frac{1}{2} \). The \( \theta \) equation for the \( E = 0 \) state can be mapped to a non-Hermitian TBM:

\[
\bar{\theta}_{m+2} + \bar{\theta}_{m-2} + 2i\Lambda_e \cos(2\pi \alpha_1 m - k_y) \bar{\theta}_m = 0, \tag{5}
\]

where \( \Lambda_e = 2\Lambda^2 \cos(k_y) \) if \( m \) is even and \( \Lambda_e = 2\Lambda^2 \sin(k_y) \) if \( m \) is odd, and \( \bar{\theta}_m = i^m \theta_m \). Analogous to the corresponding Hermitian problem, the system exhibits self-duality at \( \Lambda_e = 1 \), and this self-dual point describes the onset of localization [12]. Therefore, the boundary curve describing the localization transition for the \( E = 0 \) state corresponds to the minimum value of \( \Lambda \) consistent with the solutions of \( 2\Lambda^2 \cos(k_y) = 1 \) and \( 2\Lambda^2 \sin(k_y) = 1 \). It should be noted that for \( k_y/2\pi = \frac{1}{8} \) (mod \( \frac{1}{2} \)), even-odd boundary curves are identical and we see the emergence of a characteristic wave number that corresponds to the maximum value of \( \Lambda \) for localization of the \( E = 0 \) state. These analytic predictions are in perfect agreement with the numerical computation of the localization boundary (see Fig. 3).

The above mapping elucidates the importance of the wave number \( k_y \) in the localization transition in the non-Abelian case. The transition threshold is a sinusoidal function of \( k_y \), and there exists a special set of wave numbers that survive localization the longest as we increase the parameter \( \Lambda \). The importance of \( k_y \) and the existence of char-

![FIG. 2 (color online). Upper panel shows minimum energy for \( \alpha_2 = \frac{1}{2} \) with \( \Lambda = 0.4 \). Short and long dashed lines show the corresponding Abelian case with \( \alpha = \frac{\pi}{144} \) and \( \frac{1}{2} \), respectively. Lower panel illustrates the localization of the ground state (\( \frac{k_y}{2\pi} = \frac{1}{7} \)); localization length \( \xi \) (in units of \( \alpha \)) that with \( \Lambda = 0.4 \) (solid line) is \( \infty \) becomes finite at \( \Lambda = 0.6 \) (dashed line). Here \( E \) is treated as a parameter: the extended states correspond to \( 1/\xi \to 0 \), spikes describe localized states, and the smooth lobes correspond to forbidden regions.](image1)

![FIG. 3 (color online). Shaded region shows the extended phase for the \( E = 0 \) state for \( \alpha_2 = \alpha_1^\alpha \). Long and short lines show the boundary of the extended phase for \( \alpha_2 = \frac{1}{2} \) and \( \alpha_2 = \frac{1}{4} \), respectively. Solid vertical line shows the localization boundary for the Abelian problem described by Eq. (1) with \( \alpha = \) golden mean.](image2)
acteristic wave numbers illustrated explicitly for $E = 0$ state is also valid for ground states as it is precisely at these wave numbers that the full energy spectrum for even and odd sites for $\theta$ (or $\eta$) are fully compatible. This special set of wave numbers also corresponds to the ground state of the system. Interestingly, these features also characterize the localization transition for irrational $\alpha$ and suggests the theoretical results are universal with respect to the choice of $\alpha$. The localization boundary for $\alpha = \alpha^4_0$ and $E = 0$ (see Fig. 3) clearly shows the emergence of a selected set of wave numbers $k_n/2\pi = \alpha^4_0/n$ ($n = 0, 1, 2, \ldots$) that remain extended for the maximum range of $\Lambda$. Our detailed analysis of this case shows that the localization begins at $\Lambda = 0.14$ and all states are localized before $\Lambda = 0.7$.

We now turn our attention to the experimental aspects of creating and observing the effects of non-Abelian potentials. A proposed experimental implementation of artificial Abelian [1] and non-Abelian fields [5] consists of a two-dimensional optical lattice populated with cold atoms that occupy two hyperfine states. The typical kinetic energy tunneling along the $x$ direction is suppressed by accelerating the system or applying an inhomogeneous electric field in that direction. Tunneling is accomplished instead with laser driven Raman transitions, induced by additional running wave lasers detuned to cancel the effect of the acceleration. To generate non-Abelian fields, we have two sets of Raman transitions.

We can experimentally access various $\Lambda$ values (ratio of the tunneling in $y$ and $x$ directions) by adjusting the lattice beam intensity. Explicit formulas for tunneling due to kinetic energy ($J$) [13] and numerical calculations of laser-induced tunneling have been reported [1]. Based on those results, one can write $\Lambda/E_R = (1/\hbar \Omega) f(V_0/E_R, \alpha)$, where $f$ is a numerically known function. Here $V_0$ is the optical potential of the lattice, $E_R$ is the lattice recoil energy, and $\Omega$ is the Rabi frequency associated with laser-induced tunneling. In the non-Abelian case, there are generally two possible values of $\Lambda$ corresponding to $\Omega_1$ and $\Omega_2$. By adjusting $\Omega_2/\Omega_1 = f(V_0/E_R, \alpha_1)/f(V_0/E_R, \alpha_2)$, we obtain a single $\Lambda$ in correspondence with the theoretical studies described here. For a practical range of $V_0/E_R$ between 5 and 45, $f$ decreases monotonically from 0.5 to nearly zero. To create a useful parameter range to see the metal-insulator transition ($0 < \Lambda < 2$), we require that the parameter $\hbar \Omega / E_R$ be set to order unity. It is possible to achieve this with reasonable experimental settings for alkali atoms like $^{87}$Rb if the two states are taken to be the hyperfine levels of the $^2S_1/2$ state.

In cold atom optics, metal-insulator transitions can be detected by measuring the momentum distribution of the expanding atomic clouds. The Bragg peaks in the metallic phase (as shown in Fig. 4), reflecting the quasiperiodic order, become flattened at the onset to localization. It should be noted that in the non-Abelian case, the momentum distribution shows additional satellite peaks around the Bloch vectors. Finally, it should be noted that in an optical lattice with a finite number of sites, the signature of the localization transition can be observed in a controlled manner by tuning $\alpha$ to a sequence of rational approximants to the golden mean.

We would like to emphasize that the artificial gauge fields discussed here are not physical. Therefore, although formally analogous to a system with non-Abelian gauge fields, the model under investigation here is a system with internal symmetry that is spontaneously broken. Such a $U(2)$ gauge system clearly offers a much richer environment than does $U(1)$. For example, using polarized fermions such as $^{40}$K with the Fermi energy in the central band, one may observe the complete deliance of localization. For a Bose-Einstein condensate loaded in the lattice, smooth changes in the ground state energy for the Abelian system will be replaced by extremely sharp oscillations as the transverse component of the momentum is tuned (see Fig. 1). In addition to controlling the nature of the quantum phases by the ratio of the tunneling in two directions, the transport properties can also be controlled by the momentum. Such novelities will broaden the possibilities of using ultracold gases in exploring new horizons in physics.