Eigenfunctions \( v_{km} \) labeled by an energy eigenvalue \( E \), and an orbital angular momentum quantum number \( l \) are still degenerate with respect to the magnetic quantum number \( m \). However the operator associated with \( m, L_z = -i\hbar(\partial/\partial \phi) \), is pure imaginary and, in accordance with Eq. (29.16), does not commute with \( K \). Thus we cannot argue, as was done in the preceding paragraph, that \( K v_{km} = c v_{km} \) so that \( v_{km} \) is real; in actuality it is not real in general since it is proportional to the spherical harmonic \( Y_{lm}(\theta, \phi) \), which is complex for \( m \neq 0 \).

If, however, we restrict our attention to the case \( m = 0 \), we are dealing only with eigenfunctions of \( L_z \) that have the eigenvalue zero. It then follows from the relation \( L_z K = -KL_z \), that \( K \) times an eigenfunction of \( L_z \) is still an eigenfunction of \( L_z \) with eigenvalue zero. Thus if there is no additional degeneracy, we can be sure that those eigenfunctions \( v_{k0} \) of \( H \) and \( L^2 \) that have \( m = 0 \) are all real in the sense used above. These are just the eigenfunctions used in the scattering theory of Secs. 19 and 20. As expected, the phase shifts \( \delta_l \) are real or complex according as the hamiltonian is or is not time-reversal invariant.\(^1\)

30. DYNAMICAL SYMMETRY

We have seen in Sec. 26 that symmetry and degeneracy are associated with each other. \( \) For example, a system that possesses space-displacement symmetry is usually degenerate with respect to the direction of the momentum vector \( p \), an exception arising when \( p = 0 \). Similarly, a system that possesses rotational symmetry is usually degenerate with respect to the direction of the angular momentum vector \( J \), that is, with respect to the eigenvalue of a particular component such as \( J_z \). Again, the case \( J = 0 \) is exceptional. In the cases of the discrete symmetries of space inversion and time reversal, degeneracy is less common, since the transformed states are more likely to be the same as the original states.

It was pointed out in Chap. 4 that the hydrogen atom (Sec. 16) and the isotropic harmonic oscillator (Prob. 12) have additional degeneracy beyond that associated with rotational symmetry. As remarked there, this is to be expected whenever the wave equation can be solved in more than one way, in different coordinate systems or in a single coordinate system oriented in different ways. From our present point of view, we also expect these degeneracies to be associated with some symmetry, which evidently is not of the geometrical type considered thus far in this chapter. We call such symmetries dynamical, since they arise from particular forms of the force law. In the two relatively simple cases con-

\(^1\) For an extension of the methods of this subsection to systems in which spin is significant, see R. G. Sachs, "Nuclear Theory," App. 3 (Addison-Wesley, Reading, Mass., 1953).

considered in this section, the existence and general nature of the dynamical symmetry can be inferred from the corresponding classical system, in much the same way as with geometrical symmetries. This is not possible in general; indeed, many situations of physical interest have no classical analogs.

CLASSICAL KEPLER PROBLEM

The classical hamiltonian for the Kepler problem in relative coordinates is

\[
H = \frac{p^2}{2\mu} - \frac{\kappa}{r}
\]

(30.1)

where \( \mu \) is the reduced mass and \( \kappa \) is a positive quantity. For the hydrogen atom, comparison with Sec. 16 shows that \( \kappa = Ze^2 \). A particular solution of the classical orbit problem is an ellipse with semimajor axis \( a \) that is equal to half the distance from perihelion \( P \) to aphelion \( A \) (Fig. 25), and with eccentricity \( e \) that is equal to \((a^2 - b^2)/a \), where \( b \) is the semiminor axis.

Since \( H \) is independent of the time, the total energy \( E \) is a constant of the motion. Also, since \( H \) possesses rotational symmetry, the orbital angular momentum \( L = r \times p \) is a constant of the motion. Both of these statements are easily established from Eq. (24.22) and require calculation of some Poisson brackets. It is not difficult to show that

\[
E = -\frac{\kappa}{2a} \quad \text{and} \quad L^2 = \mu a (1 - e^2)
\]

(30.2)

\( L \) is evidently an axial vector that is perpendicular to the plane of the orbit. The rotational symmetry of \( H \) is enough to cause the orbit to lie in some plane through \( O \), but it is not enough to require the orbit to be closed. A small deviation of the potential energy from the newtonian form \( V(r) = -\kappa/r \) causes the major axis \( PA \) of the ellipse to precess slowly, so that the orbit is not closed. This suggests that there is some quantity, other than \( H \) and \( L \), that is a constant of the motion and that can be used to characterize the orientation of the major axis in the orbital plane. We

Fig. 25 Classical Kepler orbit with the center of attraction at a focus \( O \) of the ellipse.
Thus look for a constant vector $\mathbf{M}$, which we expect to lie along the major axis, pointing from $O$ to $P$ or from $O$ to $A$.

Such a vector has been known for a long time and is called the Lenz vector or the Runge-Lenz vector.¹ We write it in the form

$$\mathbf{M} = \frac{\mathbf{p} \times \mathbf{L}}{\mu} - \frac{r}{\kappa} \mathbf{r}$$

(30.3)

It is easily seen to be a constant of the motion, to have magnitude $\kappa r$, and to be directed from $O$ to $P$. The following relations are independent of the particular choice of the orbital parameters $a$ and $\epsilon$:

$$\mathbf{L} \cdot \mathbf{M} = 0 \quad \mathbf{M}^2 = \frac{2H}{\mu} \mathbf{L}^2 + \kappa^2$$

(30.4)

**HYDROGEN ATOM**

In order to treat the hydrogen atom, the foregoing quantities must be translated into quantum mechanics. This has already been done for $\mathbf{r}$, $\mathbf{p}$, and $\mathbf{L}$. For $\mathbf{M}$, we note that $\mathbf{p} \times \mathbf{L}$ is not equal to $-\mathbf{L} \times \mathbf{p}$, so that Eq. (30.3) does not define a hermitian quantity. We therefore redefine $\mathbf{M}$ as a symmetric average:

$$\mathbf{M} = \frac{1}{2\mu} (\mathbf{p} \times \mathbf{L} - \mathbf{L} \times \mathbf{p}) - \frac{r}{\kappa} \mathbf{r}$$

(30.5)

It can then be shown from the commutation relations for $\mathbf{r}$ and $\mathbf{p}$, after a considerable amount of computation, that

$$[\mathbf{M}, H] = 0 \quad \mathbf{L} \cdot \mathbf{M} = \mathbf{M} \cdot \mathbf{L} = 0$$

$$\mathbf{M}^2 = \frac{2H}{\mu} (\mathbf{L}^2 + \hbar^2) + \kappa^2$$

(30.6)

These are the quantum-mechanical analogs of the constancy of $\mathbf{M}$ and of Eqs. (30.4).

Equations (30.5) and (30.6) were used by Pauli² to find the energy levels of the hydrogen atom, independently of and simultaneously with Schrödinger's³ treatment on the basis of the wave equation that was discussed in Sec. 16. Pauli's approach is equivalent to regarding the three components of $\mathbf{M}$ as generators of some infinitesimal transformations, in much the same way that the three components of $\mathbf{L}$ were regarded in Sec. 27 as generators of infinitesimal rotations about the three axes. We thus proceed by working out the algebra of the six generators $\mathbf{L}$, $\mathbf{M}$, which consists of 15 commutation relations. Three of these have already been

² W. Pauli, Z. Physik 36, 336 (1926).
It is easily verified that Eqs. (30.12) and (30.13) lead to the commutation relations (30.7), (30.8), and (30.11).

The six generators $L_{ij}$ obviously constitute the generalization of the three generators $L$ from three to four dimensions. The group that they generate can be shown to be the proper rotation group or orthogonal group in four dimensions, designated $O(4)$, which is the set of all $4 \times 4$ real orthonormal matrices with determinant equal to $+1$. This evidently does not represent a geometrical symmetry of the hydrogen atom, since the fourth component $r_4$ and $p_4$ are fictitious and cannot be identified with dynamical variables. For this reason, $O(4)$ is said to describe a dynamical symmetry of the hydrogen atom. It does, of course, contain the geometrical symmetry $O(3)$ as a subgroup.

It is important to note that the $O(4)$ generators were obtained by restricting our considerations to bound states. For continuum states, $E$ is positive, and the sign inside the square root of (30.10) must be changed in order for $M'$ to be hermitian. Then the sign on the right side of Eq. (30.11) is changed, and the identifications (30.12) and (30.13) are no longer valid. It turns out that the dynamical symmetry group in this case is isomorphic to the group of Lorentz transformations in one time and three space dimensions, rather than to the group of rotations in four space dimensions.¹

**ENERGY LEVELS OF HYDROGEN**

The energy eigenvalues can now be found with practically no further effort. We define two quantities

\[
I = \frac{i}{2}(L + M') \quad K = \frac{i}{2}(L - M')
\]

(30.14)

which are easily seen to satisfy the commutation relations

\[
[I_x, I_y] = i\hbar I_z \quad \text{etc.} \quad [K_x, K_y] = i\hbar K_z \quad \text{etc.}
\]

\[
[I, K] = 0 \quad [I, H] = [K, H] = 0
\]

(30.15)

Thus $I$ and $K$ each constitute an $O(3)$ or $SU(2)$ algebra, and we see at once that the possible eigenvalues are

\[
I^2 = i(k + 1)\hbar^2 \quad K^2 = k(k + 1)\hbar^2 \quad i, k = 0, \frac{1}{2}, 1, \ldots
\]

(30.16)

It is easily seen from the commutation relations (30.15) that the $O(4)$ group is of rank 2. Thus there are two Casimir operators, which may evidently be chosen to be

\[
I^2 = \frac{i}{2}(L + M')^2 \quad \text{and} \quad K^2 = \frac{i}{2}(L - M')^2
\]


Alternatively, they may be chosen to be the sum and difference of $I^2$ and $K^2$:

\[
C = I^2 + K^2 = \frac{1}{2}(L^2 + M'^2) \quad \text{and} \quad C' = I^2 - K^2 = L \cdot M'
\]

(30.17)

The second of Eqs. (30.6) shows that $C' = 0$, so that we are dealing only with that part of $O(4)$ for which $L^2 = K^2$. Thus $i = k$, and the possible values of the first Casimir operator are

\[
C = 2k(k + 1)\hbar^2 \quad k = 0, \frac{1}{2}, 1, \ldots
\]

(30.18)

The third of Eqs. (30.6), together with (30.10) and (30.17), then gives

\[
C = \frac{1}{2} \left( L^2 - \frac{k}{2} M^2 \right) = -\frac{\mu^2}{4} - \frac{i}{2} \hbar^2
\]

With the expression (30.18) for $C$, we obtain

\[
E = -\frac{\mu^2}{2\hbar^2(2k + 1)^2}
\]

(30.19)

Equation (30.19) agrees with the wave equation result (16.15) if we remember that $z = Ze^i$ and make the natural identification $n = 2k + 1$, which gives $n$ the sequence of values $1, 2, 3, \ldots$.

It is important to note that there is no objection to using half-odd-integer values for $i$ and $k$ in (30.16). The only physical restriction is that $L^2 = (l + 1)\hbar^2$ have only integer values of $l$. But, since $L = I + K$ from (30.14), the triangle rule of Sec. 28 shows that $l$ can have any value ranging from $i + k = 2k - n - 1$ down to $|i - k| = 0$, by integer steps. Thus $l$ not only is restricted to integer values but has the correct range of values with respect to the total quantum number $n$. The degeneracy of this energy level is also given correctly since $I_x$ and $K_y$ can each have $2k + 1 = n$ independent eigenvalues, and there are therefore $n^2$ possible states altogether.

Finally we note that, as discussed in connection with Eq. (29.5), $L$ is an axial vector and does not change sign on space inversion. In similar fashion, it is apparent that $M$ defined by (30.5) is a polar vector, which does change sign. Thus we expect that states defined by the symmetry generators $L$ and $M$ need not have well-defined parity. This is actually the case, since states of even and odd $l$ are degenerate in the hydrogen atom.

**CLASSICAL ISO TRAPIC OSCILLATOR**

The three-dimensional isotropic harmonic oscillator is described by the hamiltonian

\[
H = \frac{p^2}{2m} + \frac{1}{2}Kr^2
\]

(30.20)
This is the generalization of the linear harmonic oscillator discussed in Sec. 13, for the case in which the force constant is the same in all directions. A particular solution of the classical orbit problem is an ellipse with semimajor axis $a$ and semiminor axis $b$, which has its major axis oriented so as to make an angle $\gamma$ with the $x$ axis (Fig. 26). As in the Kepler problem, $H$ and $L$ are constants of the motion, with values given by

$$E = \frac{1}{2}K(a^2 + b^2) \quad \text{and} \quad L^2 = mKa^3b^3$$

The fact that the orbit is closed again suggests that there is some other constant of the motion that can be used to characterize the orientation angle $\gamma$. There is, however, a striking difference between Figs. 25 and 26. In the Kepler problem, the center of attraction $O$ is at a focus of the ellipse, whereas in the oscillator problem it is at the center. Thus the two directions $OA$ and $OP$ along the major axis are not equivalent in the Kepler problem, and the minor axis is not a symmetry element. In contrast, both directions along the major axis and both directions along the minor axis are equally good symmetry elements in the oscillator orbit. Thus we expect that the additional constant of the motion is not a vector, as in the Kepler problem, but rather a quadrupole tensor.

We define the components of the quadrupole tensor as in the last five of Eqs. (27.44). Computation of the appropriate Poisson brackets then shows that the $Q'$s are constants if and only if we choose $\alpha$ and $\beta$ such that $\alpha/\beta = mK$. For the orbit shown in Fig. 26, the $Q'$s then have the values

$$Q_{xx} = \frac{1}{2}a(a^2 - b^2) \sin 2\gamma \quad Q_{xy} = Q_{yx} = 0$$

$$Q_{x} = \frac{a}{2\sqrt{3}}(a^2 + b^2) \quad Q_{y} = \frac{1}{2}a(a^2 - b^2) \cos 2\gamma$$

As expected from Fig. 26, the components of the quadrupole tensor are unchanged if $\gamma$ is replaced by $\gamma + \pi$, and also if $a$ and $b$ are interchanged and $\gamma$ is replaced by $\gamma \pm \frac{1}{2}\pi$. 

**Fig. 26** Classical harmonic-oscillator orbit with the center of attraction at the center $O$ of the ellipse.

**Quantum Isotropic Oscillator**

Since the quantum problem separates in cartesian coordinates, the solution is easily found in terms of those given in Sec. 13. The energy levels are

$$E_n = (n + \frac{1}{2})\hbar (\frac{K}{m})^{\frac{1}{2}} \quad n = n_x + n_y + n_z$$

$$n_x, n_y, n_z = 0, 1, 2, \ldots$$

(30.21)

The degeneracy of $E_n$ is easily seen to be $\frac{1}{2}(n + 1)(n + 2)$, and the part of this state is even or odd according as $n$ is even or odd. Thus the only possible $l$ values are $n, n - 2, \ldots$ down to 1 or 0, and it can be shown that each $l$ occurs just once.

Comparison with the work of Sec. 27 shows that the dynamical symmetry group is $SU(3)$. Since we require that $\alpha/\beta = mK$, the Casimir operator (27.47) is related to the square of the Hamiltonian (30.20):

$$C = -3 + \frac{4m}{3\hbar^2K}H^2$$

(30.22)

Substitution of the expression (30.21) for the $n$th eigenvalue of $H$ in (30.22) gives

$$C = \frac{1}{3}(n^2 + 3n)$$

Since $SU(3)$ is of rank 2, there are two Casimir operators; they can be expressed in terms of two parameters, $\lambda$ and $\mu$, which take on the values 0, 1, 2, 3, 4, 5. The general expression for the first Casimir operator (27.47) in terms of these parameters is

$$C = \frac{1}{3}(2\lambda + \lambda^2 + 2\mu - 3\lambda + 3\mu)$$

Thus only the representations of $SU(3)$ with $(\lambda, \mu) = (n, 0)$ are realized by the isotropic oscillator. The situation here is somewhat analogous to that in the hydrogen atom, where only the representations of $O(4)$ with $i = k$ are realized.

In contrast with the hydrogen atom, we have seen that there is no parity mixing in the isotropic oscillator, since the $l$ values in each degenerate state are either all even or all odd. This is to be expected since a few of the generators, the three components of $L$ and the five $Q$'s, are left unchanged in sign on space inversion.

The connection between the isotropic oscillator and $SU(3)$ can also be made through the raising and lowering operators, $a'$ and $a$, defined in Sec. 25. There are now three of each, one pair associated with each of the three coordinates. The commutation relation (25.10) and the 


2. Lipkin, op. cit., p. 126; his definition of $C$ differs from that used here by a factor of 1/3.
hamiltonian (25.11) become

\[ [a_i, a_j^\dagger] = \delta_{ij} \quad \text{and} \quad H = \left( \sum_{i=1}^{3} a_i^\dagger a_i + \frac{3}{2} \right) \hbar \omega \]

where \( i, j = x, y, z \). Then the algebra of the nine operators \( a_i^\dagger a_j \) is that of the generators of \( U(3) \); linear combinations of these can be found that are equal to \( H \) and the eight generators of \( SU(3) \).

PROBLEMS

1. Show that Eq. (26.6) is valid, making use of the form (26.4) for \( U_r(\phi) \) and the commutation relations for the components of \( r \) and \( p \).
2. Make use of the invariance of the scalar product of any two vectors under rotations, in order to show that the rows and columns of the rotation matrix \( R \) are respectively orthonormal to each other. Show also that the transpose of \( R \) is equal to the inverse of \( R \) and that the determinant of \( R \) is equal to \( \pm 1 \).
3. Show that the three commutation relations (27.14) are valid, making use of the form (27.7) for \( L \) and the commutation relations for the components of \( r \) and \( p \).
4. Show that the three matrices \( S \) defined in Eqs. (27.11) satisfy the relations \( S \times S = i \hbar S \). Show also that \( S^2 = 2 \hbar^4 \).
5. Show that the matrix elements of \( r \) for states that are rotated through the infinitesimal vector \( \phi \) are equal to the corresponding matrix elements of \( r_L = r + \phi \times r \) for the original states.
6. Show that the matrix elements of \( J \) for states that are rotated through the infinitesimal vector \( \phi \) are equal to the corresponding matrix elements of \( J_L = J + \phi \times J \) for the original states.
7. Show that the eigenvalues of \( S_\theta \), given in (27.11) are the same as those of \( J_\theta \), given in (27.26) for \( j = 1 \). Then find the most general unitary matrix that transforms \( S_\theta \) into \( J_\theta \) : \( US_\theta U^\dagger = J_\theta \). Choose the arbitrary parameters in \( U \) so that it also transforms \( S_\theta \) into \( J_\theta \) and \( S_\phi \) into \( J_\phi \). Into what does this \( U \) transform the vector wave function \( \psi_\theta \) of Eq. (27.5)?
8. Establish Eq. (27.28) by using the definitions (27.27) and the properties of the spherical harmonics given in Sec. 14.
9. Obtain an explicit expression for \( U_\theta(\phi) = \exp (-i \phi \cdot \mathbf{J}/\hbar) \) in the form of a \( 2 \times 2 \) matrix when \( J \) is given by Eq. (27.26) with \( j = \frac{1}{2} \). Let the vector \( \phi \) have the magnitude \( \phi \) and the polar angles \( \theta \) and \( \phi \). Show explicitly that your matrix for \( U_\theta(\phi) \) is unitary and that it is equal to \( -1 \) when \( \phi = 2 \pi \).
10. Show that the matrices \( \lambda_j \) \( (j = 1, \ldots, 8) \) defined by Eqs. (27.37) and (27.40) satisfy the commutation relations (27.41) and (27.42). Then use these commutation relations (not the original matrix representation of the \( \lambda_j \)) to show that each \( \lambda_j \) commutes with the Casimir operator \( C \) defined by Eq. (27.43).
11. Show that the 28 commutators of the eight operators (27.44), computed from the commutation relations between the components of \( r \) and \( p \), agree with the commutators of the \( \lambda_j \) when the identifications (27.45) and (27.46) are adopted.