Lecture II: Theoretical Physics (Phys701)

In last lecture, we rewrote Maxwell’s equations in a covariant form.

Why??
(1) Exercise in Tensor algebra
(2) Scientists demand that laws of physics must be covariant under,
(a) space and time translation, Galilean transformation
(b) Rotations in real 3D space
(c) Lorentz transformation, generalization of Galilean transformation when time is not absolute.

Galilean Transformation

Transformations between two inertial frames moving with velocity \( v_0 \) along z-axis are,

\[
\begin{align*}
x' &= x_1 \\
y' &= y_1 \\
z' &= z_1 - v_0 t \\
t' &= t
\end{align*}
\]

Note that Newtons laws (\( m\ddot{a} = \vec{F} \)) of motion are covariant under these transformations:

\[
\begin{align*}
\vec{a}' &= \vec{a} \\
\vec{v}' &= \vec{v} - \vec{v}_0
\end{align*}
\]

Covariant form of Maxwell’s equations show that these equations are covariant under Lorentz transformation,

\[
\begin{align*}
x'_1 &= x_1 \\
x'_2 &= x_2 \\
x'_3 &= \gamma(x_3 - i\beta x_4) \\
x'_4 &= \gamma(x_4 + i\beta x_3)
\end{align*}
\]

where \( \beta = v_0/c \) and \( \gamma = 1/\sqrt{1 - v_0^2/c^2} \).
Lorenz equations can be written as,

\[ x_\mu = a_{\mu\nu}x_\nu \]

Therefore, in Minkowski space, any vector should transform same as \( x_\mu \). This gives us the transformation for the field tensor, \( f_{\mu\nu} \),

\[ F'_{\mu\nu} = a_{\mu\alpha}a_{\nu\beta}F_{\alpha\beta} \]

( Note, we are using the convention that repeated index is summed over ).

----- Home Work

Show that \( c^2B^2 - E^2 \) is a scalar under Lorentz transformation.
1 Curvilinear Coordinates

In the above discussion, we had restricted to Cartesian coordinates.

Why study Curvilinear Coordinates???

Cartesian coordinate system offers a unique advantage that the three unit vectors, \((\mathbf{i}, \mathbf{j}, \mathbf{k})\) are constants in magnitude as well as in direction. However, not all problems are well adopted to the solution in Cartesian coordinates. Example, central force problem.

In solving physics problems, we are primarily interested in coordinates in which the equation can be solved easily.

Orthogonal Coordinates

In rectangular coordinates, we deal with three mutually perpendicular families of planes: \(x_i = \text{constant} \ (x = x_1, y = x_2 \text{ and } z = x_3).\) For general orthogonal curvilinear coordinates \(q_i,\) we deal with three orthogonal surfaces which need not be planes.

\[
ds^2 = \sum dx_i^2 = \sum g_{ij} dq_i dq_j = \sum h_i dq_i^2\]

Collectively, the coefficients \(g_{ij}\) are referred to as metric. For orthogonal systems, \(g_{ij} = 0\) for \(i \neq j.\) \(h_i\) are scale factors. Since \(q_i\) need not have the dimension of length, \(h_i dq_i\) have the dimension of length.

Review familiar Coordinate systems

NO notes: Read from any book

——— Home Work

A particle is moving through space. Find components of the velocity and acceleration in (a) circular cylindrical (b) spherical polar coordinates.

NOTE: You can use Lagrange equations or simply differentiate \(\mathbf{r}(t),\) expressed in terms of cylindrical and spherical coordinates.

Fields

Practically all of Modern physics (beyond Newtonian mechanics) deals with Fields.

Mathematically speaking, a field is a set of functions of coordinates of a point in space. It cannot be analyzed in terms of the positions of finite number of particles.

For example, most of the problems involve the equation for the field \(\psi,\)

\[(\nabla^2 + k^2)\psi = 0\]  \hspace{1cm} (2)

Note, \(k^2 = 0\) is Laplace equation
$k^2$ negative is the diffusion equation
$k^2$ equal to a constant times kinetic energy describes Schrödinger equation

It has been shown that (Phys Rev 45, 427, 1937 by Eisenhart) this equation is separable in 11 different coordinate systems. In spite of the fact that $\nabla^2$ is very complicated in non-Cartesian coordinates, one needs to use coordinate system in which partial differential equation is separable.

2 Separability in more than one coordinate system

Examples
(1) H-atom can be solved in spherical as well as parabolic coordinate systems
(2) Three-dimensional isotropic harmonic oscillator can be solved in rectangular and spherical coordinates.
We will explore in detail the relationship between symmetry and separability in more than one coordinate system.

3 H-Atom: Symmetry, Separability, Conserved Quantity and Degeneracy

CLASSICAL Problem
Using Lagrange equations of motion, show that spherical symmetry implies that angular momentum is conserved and motion is in the plane.

For $V(r) = 1/r, 1/r^2, r^2$ types of potentials there exists another constant of motion.
For these special class of potentials, the Schrödinger equation is separable in more than one coordinate system. For example, for H-atom, the equation is also separable in parabolic coordinates: $\xi, \eta, z$, where,

$$
\begin{align*}
\xi &= x - z \\
\eta &= x + z \\
z &= z
\end{align*}
$$
Consider H-atom

In atomic units $\hbar = 1 = e = m$, this new constant is called Runge-Lenz vector,

$$ N = pXL - \hat{r} \quad (3) $$

The corresponding symmetry is called dynamical symmetry as this rather subtle symmetry is clearest in the momentum space. The existence of this constant of motion is responsible for bounded orbits (non-precessing orbits).

In Quantum mechanics, $[H, L] = 0$. This means that energy is independent of the quantum number $m$.

Also, $[H, N] = 0$ where $N$ is defined as,

$$ N = (pXL - LXp)/2 - \hat{r} \quad (4) $$

Just as $L$ are generators of rotation and generate $O(3)$ group, $L$ and $M$ together can be shown to generate $O(4)$ group, that is, rotation in 4-dimension. We will revisit this $O(4)$ group when we discuss group theory.

Using the properties of these operators, one can obtain energy levels of H-atom, without solving differential equation.

All discussion related to accidental degeneracy of the H-atom in the American Journal of Physics paper, posted on the web.

Also, various details of the Lenz vector can be seen in the google link, posted on the course web site.

4 Harmonic Oscillator

$$ H = p^2/2m + 1/2Kr^2 $$

Three-dimensional harmonic oscillator can be solved in rectangular, spherical, cylindrical as well as in spheroidal coordinate system.

The energy of a classical oscillator $E = 1/2K(a^2 + b^2)$ and $L^2 = mKa^2b^2$. Here $a$ and $b$ are the semi-major and the semi-minor axes.

The fact that the orbit is closed and the semi-major axis is fixed suggests that there exists
another constant of motion that characterizes the orientation of the major axis.
In H-atom, the center of attraction is at the focus of the ellipse and therefore, both the
major and the minor axes are not symmetry axes. However in harmonic oscillator, the
force center is the center of the ellipse and therefore, both the major and the minor axes are
symmetry axes. Therefore, we expect that the additional constant of motion is not a vector,
but perhaps a tensor.