Lecture I

Quantum-Mechanical Way of Thinking

To cultivate QM way of thinking, we will not start with the fascinating historical approach, but instead begin with one of the most important expt, that sends a shock wave to those not used to QM

HISTORY

The Stern-Gerlach (SG) experiment was performed in Frankfurt, Germany in 1922 by Otto Stern and Walther Gerlach. At the time, Stern was an assistant to Max Born at the University of Frankfurt’s Institute for Theoretical Physics, and Gerlach was an assistant at the same university’s Institute for Experimental Physics. See the attached physics today article and the link to Goudsmit lecture on the discovery of electron-spin.

In 1927, T.E. Phipps and J.B. Taylor reproduced the effect using hydrogen atoms in their ground state, thereby eliminating any doubts that may have been caused by the use of silver atoms. (In 1926 the non-relativistic Schrodinger equation had incorrectly predicted the magnetic moment of hydrogen to be zero in its ground state. To correct this problem Pauli introduced ”by hand” so to speak, the 3 spin matrices which now bear his name, but which were then later shown by Dirac in 1928 to be intrinsic in his relativistic equation.)

History of Spin-1/2

(1) Magnetic Moment of an Electron???

Let us go back to what you knew in high school, or even earlier....

That is, electron is a charged particles that goes around the nucleus just the way planets go around the sun. Moving charge is equivalent to a current \( I = \frac{e}{T} = \frac{e
u}{(2\pi r)} \).

Current \( I \) in a loop of area \( A \), acts like a magnet of dipole moment \( \mu_l \):
(2) Behavior of a Magnetic Dipole Moment in a Magnetic Field

A stationary dipole subjected to magnetic field $B$ precesses about the field, with precession frequency $\omega = \frac{\mu_l}{\mu_B} \hbar$. (Torque is perpendicular to the angular momentum and hence can only change the direction of the angular momentum).

\[
\mu_l = IA \\
= ev(\pi r^2)/(2\pi r) \\
= evr/2 \\
= \frac{e}{2m}L \\
= g_l(eh/2m)(L/h) \\
= g_l\sqrt{l(l+1)} \\
\mu_l^2 = g_l\mu_B \hbar
\]

\[
E = -\mu_l \cdot B \\
\tau = \frac{dL}{dt} \\
\tau = \mu_l \times B \\
= -\frac{g_l\mu_B}{\hbar}L \times B \\
= -(\frac{g_l\mu_B}{\hbar}B)L \sin \theta \\
= \omega L \sin \theta
\]

(3) Effect of Inhomogeneous Field on a dipole $\mu$
If the magnetic field is not uniform, the dipole will experience a force, in addition to torque:
Let \( B(x, y, z) = -ax \hat{i} + (B_0 + bz)\hat{k} \), where \( B_0 \) is a strong uniform field and the constants \( a, b \) describe the non-uniform part of the field. Note, \( \nabla B = 0 \). Due to Larmor precession, \( \mu \hat{r} \) oscillates rapidly and averages to zero and the net force is in the \( z \)-direction

\[
B(x, y, z) = -ax \hat{i} + (B_0 + bz)\hat{k}, \\
F = \nabla(\mu \cdot B), \\
F_z \approx \mu_\parallel \partial_z B_z
\]

**Problem:** Compute the deflection

*A beam of H-atoms at \( T = 400 \) K is sent through SG magnet of length \( L = 1 \) m. If the atoms experience a gradient of 10 Tesla/m, calculate the transverse deflection \( Z \) of the atom, due to the force on its spin dipole moment

\[
1/2mv_x^2 = 2KT, \\
v_x = \sqrt{4KT/m}, \\
t = L/v_x, \\
Z = 1/2 \frac{F_z}{m} t^2, \\
F_z = \partial_z B \mu
\]

**Ans:** \( Z = \pm 2.1 \times 10^{-3} \) m.

**Introducing Electron Spin**
In 1927, graduate students Goudsmit and Uhlenbeck hit upon the idea of..

They did not want to publish this: but Ehrenfest told them *you both are young enough to allow yourself some foolishness*

Many came close.. Pauli exclusion principle, 1925.....

NOTE: (1) In Classical mechanics, a rigid body can have two types of angular momentum, orbital: \( L = r \times p \), associated with the motion of the center of mass and spin: \( S = I \omega \), associated with the motion about the center of mass. But in QM, distinction between these two types is very fundamental as spin has nothing to do with motion in space-time.

NOTE (2) Schrodinger QM is completely compatible with the existence of electron spin; but it does not predict it, so spin must be introduced as a separate postulate.

NOTE (3) Spin operators obey the same algebra as the angular momentum operators. However, the \( g \) factor that relates spin magnetic moment and spin angular momentum, \( \mu_s = g_s \mu_B S / \hbar \) is the \( g_s = 2 \).

**Impact of SG Experiment**

The SG experiment had one of the biggest impacts on modern physics:

(1) In the decade that followed, scientists showed using similar techniques, that the nuclei of some atoms also have quantized angular momentum. It is the interaction of this nuclear angular momentum with the spin of the electron that is responsible for the hyperfine structure of the spectroscopic lines.

(2) In the thirties, using an extended version of the SG apparatus, Isidor Rabi and colleagues showed that by using a varying magnetic field, one can force the magnetic momentum to go from one state to the other. The series of experiments culminated in 1937 when they discovered that state transitions could be induced using time varying fields or RF fields. The so called Rabi oscillation is the working mechanism for the Magnetic Resonance Imaging equipment found in hospitals.
(3) Norman F. Ramsey later modified the Rabi apparatus to increase the interaction time with the field. The extreme sensitivity due to the frequency of the radiation makes this very useful for keeping accurate time, and it is still used today in atomic clocks.

(4) In the early sixties, Ramsey and Daniel Kleppner used a SG system to produce a beam of polarized hydrogen as the source of energy for the hydrogen Maser, which is still one of the most popular atomic clocks.

(5) The direct observation of the spin is the most direct evidence of quantization in quantum mechanics.

What do we learn from SGE???

If we ignore nuclear angular magnetic moment (which is 2000 times smaller than the electronic moment) and since angular momentum quantum number \(l = 0\), the magnetic moment of the Ag atoms is solely due to the intrinsic spin of the electron. This spin cannot be described by laws of classical mechanics.

SG experiment suggests that QM States are represented by vectors in an abstract complex vector space

Note, in SG expt, only thing we care about is the spin of the atom, which way it is pointing... We do not care about the spatial coordinates of the atom..

Physical state, like Ag atom with a definite spin orientation...

Figure 1.3 suggests that this state (ie, state with the definite spin) is acting like a vector in 2D space

(2D has nothing to do with the x-y coordinates); it is a kind of abstract 2D vector space, where if we know two vectors, we can find all other vectors:

Possible states of the system are:
$|S_x, + \rangle$
$|S_x, - \rangle$
$|S_y, + \rangle$
$|S_y, - \rangle$
$|S_z, + \rangle$
$|S_z, - \rangle$

But, they are not independent: given any two, others can be determined, as suggested by Fig 1.3

Suppose, we are given $|S_z, + \rangle$ and $|S_z, - \rangle$

How can we express all other states in terms of these two alone ?? The simplest possibility is,

$$|S_x, + \rangle = A_1|S_z, + \rangle + A_2|S_z, - \rangle$$
$$|S_z, - \rangle = A_1|S_z, + \rangle - A_2|S_z, - \rangle$$

where it is obvious that $A_1 = A_2 = A = 1/\sqrt{2}$

Now, what about $S_y, \pm \rangle$ ??

The only way to make $|S_y, \pm \rangle$ independent of $|S_x, \pm \rangle$ is to use $A_1 = \pm iA_2$.

*In other words, the vector space is complex*

This is amazing, the outcome of this SG expt tell us that QM states can be represented by vectors in an abstract complex vector space.
NOTE

(1) We can choose $|S_x, \pm >$, or $|S_y, \pm >$ instead of $|S_z, \pm >$

Basic Mathematics of Vector Space used in QM

In contrast to classical mechanics where position and velocity specifies the system completely, in QM, the state of the system is represented by vectors in abstract complex vector space.

Postulates of QM + SGE determine eigenkets and also Operators

See Sakurai, Chapter I