Measurement Uncertainties Reference

Introduction

We all intuitively know that no experimental measurement can be “perfect”. It is possible to make this idea quantitative. It can be stated this way: the result of an individual measurement of some quantity is the actual value and an “error”. We associate the size of the error with the particular method of measurement—some techniques have inherently smaller errors than others (e.g. a micrometer vs. a yardstick.) We will use the terms “uncertainty” and “error” interchangeably. An important part of understanding an experiment and reporting its results is being able to determine the measurement uncertainty. A practicing scientist or engineer needs to understand measurement uncertainties both for the interpretation of measurements made by others, and for design of future measurements.

To find an uncertainty, it is necessary to find the range in which an individual measurement is likely to lie. For example, if we measure the speed of light to be 3.01 x 10^10 cm/sec, and a study of the measurement system indicates that the individual measurements are likely to lie between 2.98 and 3.04, we would quote the result as

\[ c = (3.01 \pm 0.03) \times 10^{10}\text{cm/sec} \]

The uncertainty is usually explicitly displayed with “error bars” when experimental results are graphed, as in Figure 1.2.

Random and Systematic Errors

Measurement errors fall into two categories: random and systematic. The random error in an individual measurement is not predictable in exact magnitude or sign. However, the average of random errors over many repeated, independent measurements of the same quantity is zero. Thus, the average of several measurements is generally closer to the actual value than any of the individual measurements. The uncertainty in the case of random errors should indicate the range of results that would be obtained from many measurements. This should be a characteristic of the measurement apparatus. In fact, one common method for estimating the range of random errors for a particular measuring device is simply to repeat the measurement many times. The other type of error is “systematic”. Here the magnitude and sign of the error is always the same. The measurement result is shifted, up or down, by the same amount, even in repeated measurements. Thus, averaging many measurements will not improve results. Typically a measurement has both random and systematic errors. Sometimes they can be independently estimated and quoted. If this were the case for the speed of light measurement, above, the result might be quoted:

\[ c = (3.01 \pm 0.03 \pm 0.01) \times 10^{10}\text{cm/sec} \]

Here the last term is the systematic uncertainty.

There is actually a third type of error: a mistake, i.e., an error that is caused by the person performing the measurement, not by the apparatus. These are preventable or caught with attention and some art. One of the functions of written experimental reports
is to show readers, by descriptions and internal consistency checks, that this type of error has probably been eliminated.

Systematic errors are usually unique to specific techniques, and so the discussion below will concentrate on general methods for estimating random errors.

**The RMS Deviation**

One way to specify the range of a set of values is the RMS deviation. This stands for “Root” “Mean” “Square” deviation. “Mean” in this context, means the average. Given a set of \( N \) values of \( x \), the average, or mean \( x \), denoted by \( \bar{x} \) is

\[
\bar{x} = \frac{1}{N} \sum_{i=1}^{N} x_i
\]  

(1)

For the same set of \( x \)'s we can also define the RMS deviation from the mean, denoted by \( \sigma \), through the equation

\[
\sigma = \sqrt{\frac{1}{N-1} \sum_{i=1}^{N} (x_i - \bar{x})^2}
\]  

(2)

The quantity \( \sigma \) is a measure of how dispersed are the values of \( x \) away from their average, \( \bar{x} \). If all \( x \)'s have the same value then \( \sigma \) is zero. The RMS deviation is frequently called the “standard deviation”. Strictly speaking, \( \sigma \) is the standard deviation if the \( x \)'s come from a particular kind of distribution of errors called the “normal” or “Gaussian” distribution. Another equivalent expression for \( \sigma \) in Eq (2) is given by:

\[
\sigma = \sqrt{\bar{x}^2 - \bar{x}'^2}
\]  

(3)

If the number of measurements \( N \) is increased, the values of \( x \) and \( \sigma \) tend toward limiting values which are independent of \( N \). The value of \( x \) is determined by the physical quantity being measured. The value of \( \sigma \) is determined by the measurement apparatus. It is common practice to use \( \sigma \) as the experimental uncertainty.

A more detailed study of errors involves consideration of the probability of obtaining a particular error in a measurement. One of the useful results of such a study is the following chart which gives the odds that an individual error will deviate from the mean by a specified number of standard deviations, \( N\sigma \), for the “normal” error distribution.
So it is very unlikely for an error to exceed 2-3 $\sigma$, but not so rare to exceed 1 $\sigma$. These results are useful in comparing different measurements. What matters in determining whether two measurements agree or disagree is not the absolute amount by which they differ, but the number of $\sigma$’s by which they differ. The importance of this is illustrated in the Figure 2, which displays two hypothetical measurements of the same quantity, together with a predicted value (from some theory, say.) The quantity is plotted along the vertical direction.

Experiment A deviates from the prediction by slightly more than 1 $\sigma$. It cannot be said to disagree, significantly with the prediction. It is also evident that A and B agree with each other. Experiment B, however, deviates by over 3.5 $\sigma$ from the prediction. This is a significant disagreement, even though B is closer to the prediction than A! Again, what matters is not the absolute difference, but the difference in units of uncertainty. Similarly, a calculation of the percentage deviation between two measurements, or between a measurement and a “book” value, yields a correct but meaningless number, unless the uncertainties are taken into account.
Error Estimation in Individual Measurements

As described above, one way to estimate the $\sigma$ value of a certain measurement process is to make repeated, independent measurements and compute the RMS deviation from the mean. In certain cases, simpler methods may be used. A common source of random error is the reading of a dial or scale. For such readings, a reasonable estimate of $\sigma$ is one half the smallest scale divisions. One must be careful though, many effects can cause measurement errors. For example, the digital stop watch may display time to hundredths of seconds, however the human reaction time to start and stop the watch is often 0.05 seconds or more making the timing error at least 0.05 s instead of the 0.005 s based only on the display. Alternately, some wobble in the apparatus might cause the timing of the apparatus to vary, even though one might measure time to great accuracy. Identifying the dominant source of experimental errors often requires careful reflection.

Error Propagation

Often it is necessary to combine measurements of different quantities, and to find the overall uncertainty. For example, if we measure a distance moved, $x$, during a time, $t$, for an object initially at rest with a constant applied force, then the acceleration is

$$a = (2x)/t^2.$$

If we know the uncertainties $\sigma_x$ and $\sigma_t$, what is the uncertainty in $a$, i.e. what is $\sigma_a$? The general form of this question is: given a function

$$z = f(x,y),$$

what is $\sigma_z$, given $\sigma_x$ and $\sigma_y$? For those who know calculus, the answer can be shown to be, for the case of independent measurements of $x$ and $y$, 

\begin{figure}
\centering
\includegraphics[width=0.5\textwidth]{graph.png}
\caption{Graph of Data Points A and B}
\end{figure}
Since calculus is not required for this course, we will use some formulas for special cases which are derived from the above formula:

1. **Sum or Difference:**

   If the quantity \( z \) is the sum or difference of \( x \) and \( y \), i.e. \( z = x \pm y \), then

   \[
   \sigma_z^2 = \left[ \frac{\partial f}{\partial x} \right]^2 \sigma_x^2 + \left[ \frac{\partial f}{\partial y} \right]^2 \sigma_y^2
   \]

   Note that the operation between the \( \sigma \)'s in Eq 1.4 is “+” for both summation and subtraction, so that errors always get larger (see the example below). Taking the square root of the sum of the squares is the way that the magnitudes of two vectors that are at right angles to each other add. It is called addition “in quadrature”. Thus for additions and subtractions, the errors add in quadrature. A little experimenting on the calculator or by looking at geometric vector addition will show you that in this form of addition, if one number is much larger than the other the result is a little larger than the larger number. If they are about equal, the result is about 1.4 times either one. For example: if \( a = 2.3 \text{ cm} \pm 0.3 \text{ cm} \) and \( b = 1.3 \text{ cm} \pm 0.1 \text{ cm} \), then the difference is given by:

   \[
   c = a - b = (2.3 \text{ cm} \pm 0.3 \text{ cm}) - (1.3 \text{ cm} \pm 0.1 \text{ cm})
   \]

   \[
   = 1.0 \text{ cm} \pm \sqrt{(0.3 \text{ cm})^2 + (0.1 \text{ cm})^2} = 1.0 \text{ cm} \pm 0.3 \text{ cm}
   \]

   Note that it is conventional to round to one significant figure in the uncertainties and round the number itself (not the uncertainty) to the same place as the smallest significant figure of the uncertainty (tenths in the above example).

2. **Product or quotient:**

   If the quantity \( z \) is the product or quotient of \( x \) and \( y \), i.e. \( z = xy \) or \( z = x/y \) then

   \[
   \%\sigma_z = \sqrt{\%\sigma_x^2 + \%\sigma_y^2}
   \]

   where \( \%\sigma_z = (\sigma_z/z) \times 100\% \) and therefore, \( \sigma_z = z (\%\sigma_z)/100\% \). In this form we say that for multiplication and division the percent errors add in quadrature. For example if \( v = 0.024 \text{ m/s} \pm 0.002 \text{ m/s} \) and \( t = 5.80 \text{ s} \pm 0.51 \text{ s} \), then the distance traveled is given by:

   \[
   d = vt
   \]

   \[
   = (0.024 \text{ m/s} \pm 0.002 \text{ m/s})(5.80 \text{ s} \pm 0.51 \text{ s})
   \]

   \[
   = (0.024 \text{ m/s} \pm 8\%)(5.80 \text{ s} \pm 9\%)
   \]

   \[
   = (0.024 \text{ m/s})(5.80 \text{ s}) \pm \sqrt{8\%^2 + 9\%^2} = 0.14 \text{ m} \pm 12\%
   \]

   \[
   = 0.14 \text{ m} \pm 0.02 \text{ m}
   \]
Note that we first convert the uncertainties (or errors) to percent uncertainties before adding them in quadrature. We usually convert them back to uncertainties in meters (or whatever units are being used) at the end.

3. Mix of the Above Two Types:

If a calculation has a mixture of the two types in 1 and 2 above, such as addition and division then one needs to handle the first calculation, and then the second. That is, do the addition as in 1 above, and then do the division as in 2:

\[ \nu = \frac{(B - A)}{\Delta(t)} \]
\[ = \frac{[(9.3\text{cm} \pm 0.2\text{cm}) - (3.1\text{cm} \pm 0.2\text{cm})]}{(1.8\text{s} \pm 0.4\text{s})} \]
\[ = \frac{(6.2\text{cm} \pm 0.3\text{cm})}{(1.8\text{s} \pm 0.4\text{s})} \]
\[ = (6.2\text{cm} \pm 5\%)/(1.8\text{s} \pm 22\%) \]
\[ = 3.4\text{cm/s} \pm 23\% \]
\[ = 3.4\text{cm/s} \pm 0.8\text{cm/s} \]

4. Averaging:

If the quantity \( z \) is the average of \( x \) and \( y \), i.e. \( z = (x + y)/2 \) and \( \sigma_x = \sigma_y \), then using the same procedure as above with \( \sigma_2 = 0 \), we can show that

\[ \sigma_z = \frac{\sigma_x}{\sqrt{2}} \]

The general result for averaging \( N \) independent measurements of \( x \), all with the same \( \sigma \) is

\[ \sigma_{\text{avg}} = \frac{\sigma_x}{\sqrt{N}} \] (6)

5. Multiplication by a constant:

If the quantity \( z \) is a constant value times a measurement \( x \), i.e. \( z = ax \), where 'a' is a constant with no uncertain (e.g. \( 2\pi \)) then

\[ \%\sigma_z = \%\sigma_x \] (7)

6. Taking something to a power:

If \( z = x^n \), then

\[ \%\sigma_z = n(\%\sigma_x) \] (8)

that is, the percent error of \( z \) is \( n \) times the percent error of \( x \). This holds for \( n = 2, 3, \frac{1}{2} \) (square root), and other integer and fractional exponents.
7. Special functions:

For a special function, such as sine, cosine, log, exp, then follow the procedure below:

If \( z = \sin x \) and \( x = 0.90 \pm 0.03 \), then

\[
\begin{align*}
\sin(0.93) &= 0.80 \\
\sin(0.90) &= 0.78 \\
\sin(0.87) &= 0.76
\end{align*}
\]

Therefore, \( z = \sin(0.90) = 0.78 \pm 0.02 \). Here we try various values to see how much changing the angle changes the sine of the angle.

Least Squares Linear Graph Fitting:

After data has been collected, graphical analysis of the data is desired. To interpret data, graphs are often compared to an expected shape or curve from a theoretical equation. Most likely, data collected will be graphed in such a way as to produce a straight line. This is desired since a line has a constant slope. Since we know in advance that the data will not determine a perfect straight line, we must use some technique to acquire the best straight line for the data collected. Statistical analysis provides an equation, the least squares fit (assume Gaussian population), that will minimize error and determine the best slope and \( y \)-intercept for a collection of data points \((x,y)\) for a straight line, \( y = mx + b \).

The least squares approximation for the slope is described by the equation:

\[
m = \frac{N \sum_i (x_i y_i) - \sum_j y_j \sum_k x_k}{N \sum_k x_k^2 - \sum_j x_j \sum_k x_k}
\]  \hspace{1cm} (9)

The least squares approximation for the \( y \)-intercept is described by the equation:

\[
b = \frac{\sum_i x_i^2 \sum_j y_j - \sum_i x_i \sum_j (x_i y_j)}{N \sum_i x_i^2 - \sum_j x_j \sum_k x_k}
\]  \hspace{1cm} (10)

Using the above equations an approximation for a line which best fits the data points can be determined. In Excel, the 'trendline' option in graphing with a choice of 'linear' fit will calculate the above '\( m \)' and '\( b \)' values for you and produce a line following the prescribed equations. Initially, it is a good idea to check Excel to ensure you understand what calculations it is making for you. You may have Excel display the equation for you, thus allowing you to compare these values to theoretical values of the slope and intercept.
**Percent Difference:**

To calculate the percent difference between an accepted value and the experimentally determined value, use the equation below. Remember that this value is a percentage and therefore requires a %.

\[
\% \text{ Percent Difference} = \left( \frac{\text{Accepted Value} - \text{Experimental Value}}{\text{Accepted Value}} \right) \times 100
\]

(11)