The Foundations of Special Relativity

Einstein's postulates of SR:

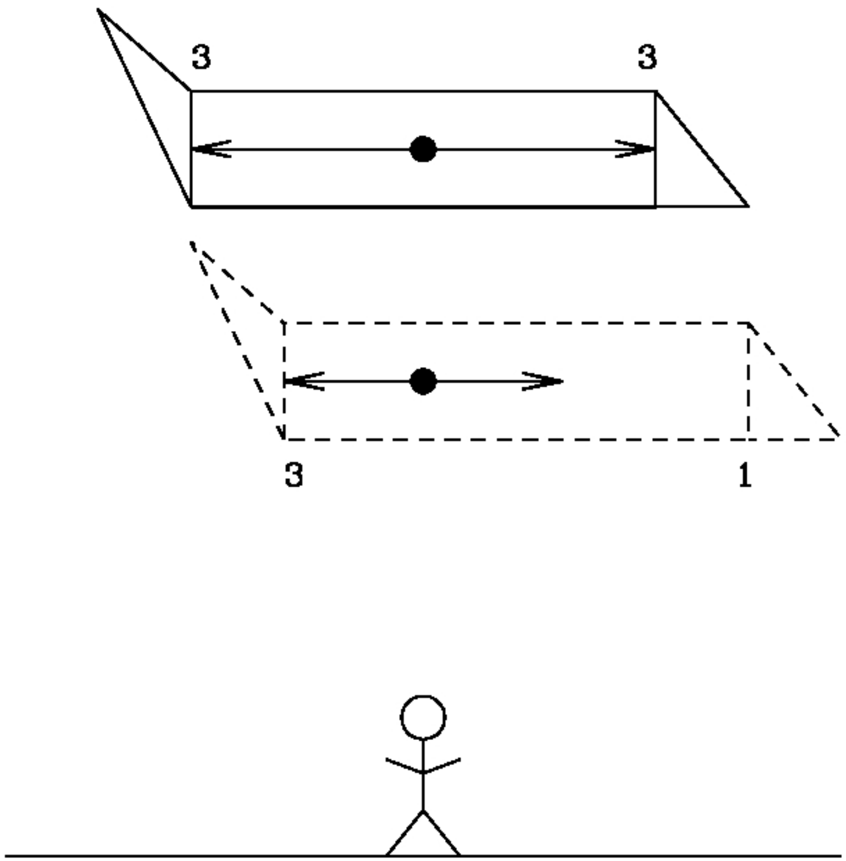
1. The laws of physics are identical in all inertial reference frames (IFs).
2. The speed of light in vacuum, c, is the same in all IFs (independent of the motion of the source or the direction of propagation).

Note: definition of IF is same as in Newtonian physics, i.e., Euclidean 3-space plus separate time and Newton's 1st Law holds.

Immediate, profound consequences:

1. Relativity of simultaneity
2. Length contraction
3. Time dilation

(see Rindler, sec 2.4)



Flash from bulb in middle of plane when middle is directly overhead for observer on ground.

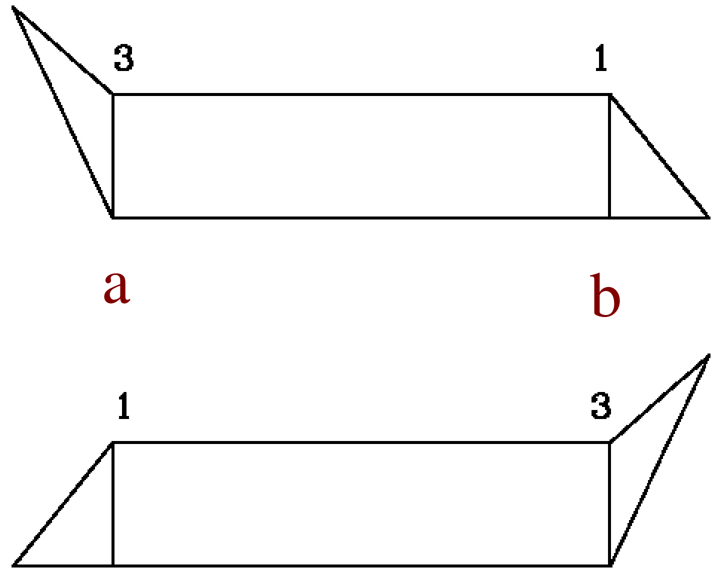
Observer in plane: light reaches the 2 ends simultaneously (say, when wristwatches at ends read $t = 3$)

Observer on ground: also sees light travel at speed c. Since back of plane moves towards bulb, light reaches back before it reaches front => relativity of simultaneity

Light striking back of plane and wristwatch at back reading "3" are the same event => observer on ground sees watch read "3" when light reaches back.

Also, light reaching front and watch there reading "3" are same event => ground observer sees watch at front read $t < 3$ when light strikes back; suppose he sees watch read "1".

Suppose another identical plane goes by in the opposite direction at the same speed.



Symmetry => ground observer sees front clock of 2nd plane also delayed by 2 units.

Consider 2 events:

a) back of top plane is lined up with front of bottom plane

b) front of top plane is lined up with back of bottom plane

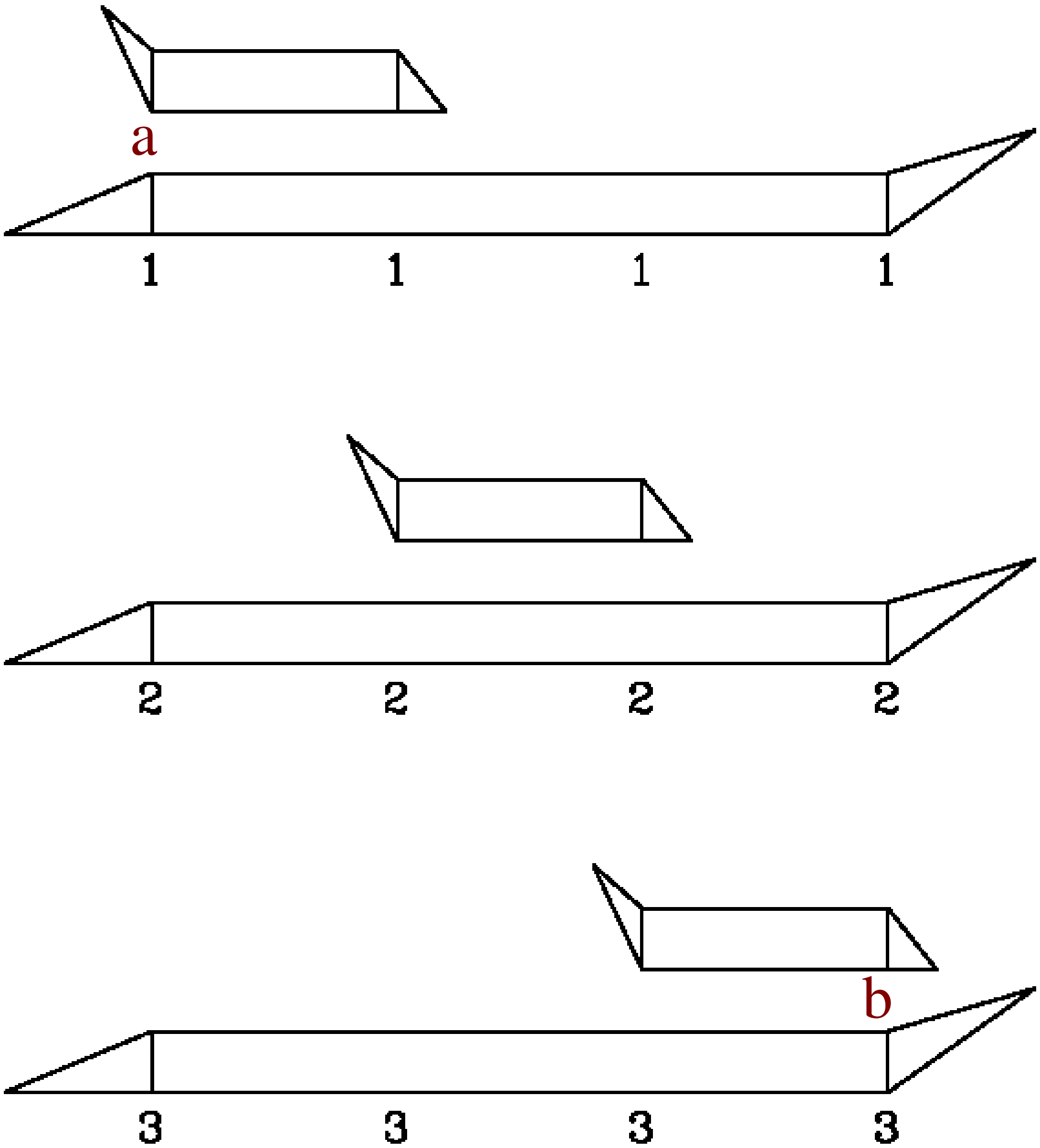
Ground observer: (a) and (b) occur simultaneously (since lengths are same)

Observer in bottom plane: (a) occurs at $t = 1$ and (b) occurs at $t = 3$

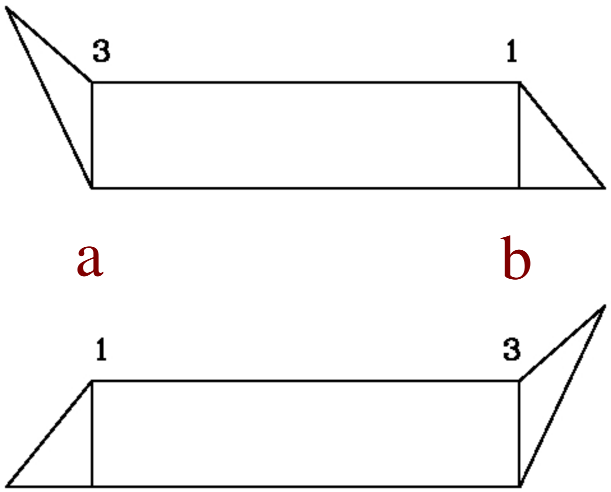
=> top plane is shorter than bottom plane

=> moving object is shortened in dir of motion (length contraction)

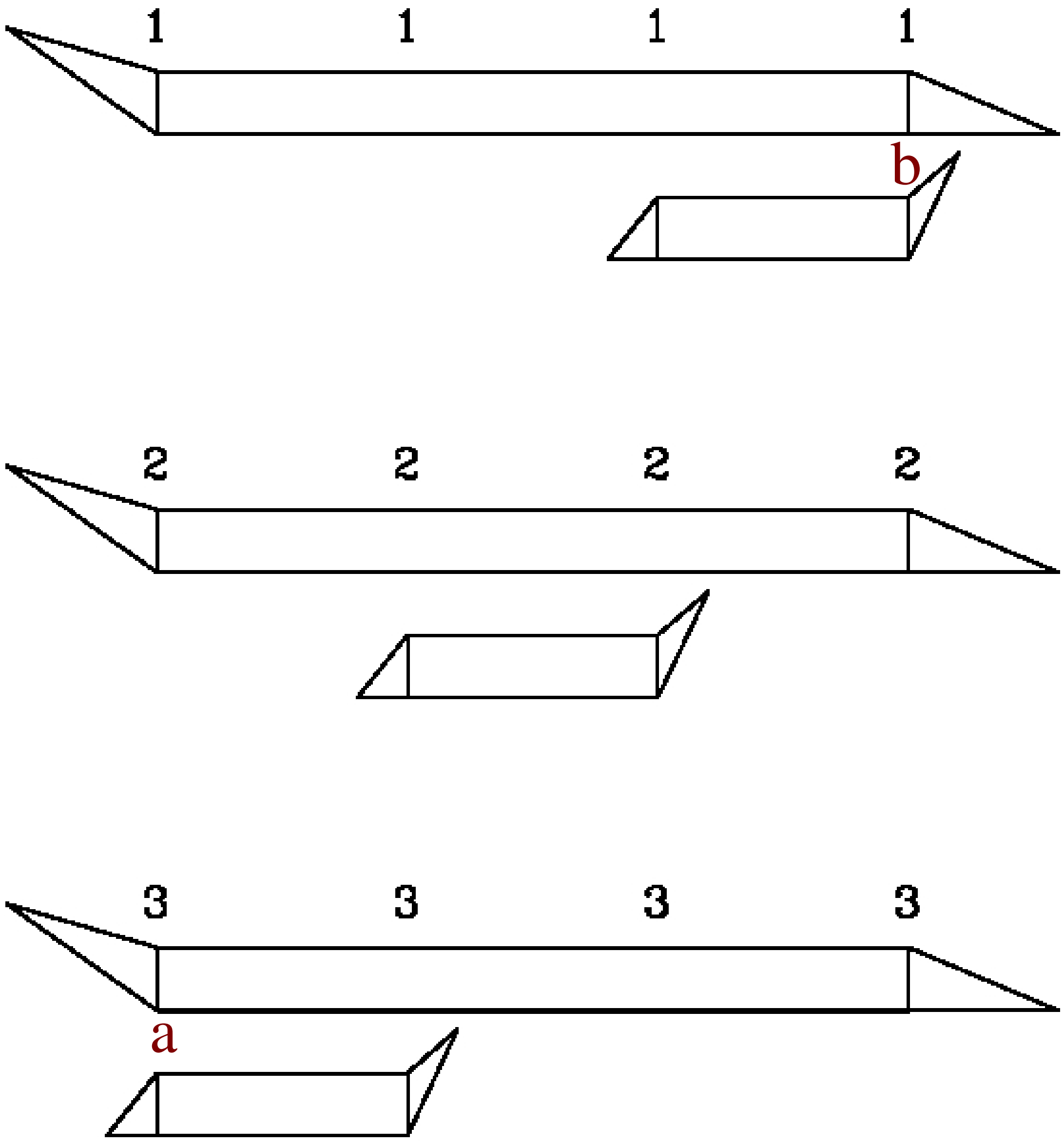
As seen by observer in bottom plane: top plane is just 1/3 the length of bottom plane.

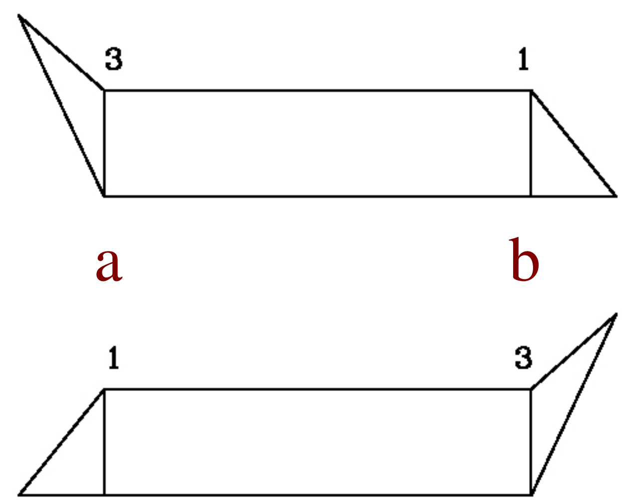


According to observer in bottom plane, all watches in the plane tick time synchronously.



Symmetrically, observer on top plane finds bottom plane is 1/3 length of top plane.





Consider clock at the back of the top plane:

Observer in bottom plane: it takes 3 units of time for that clock to move from front of bottom plane to back of bottom plane (from $t = 1$ to $t = 4$)

Observer in top plane: it only takes 1 unit of time (from $t = 3$ to $t = 4$)

=> time dilation: moving clocks run slow.

Observer in bottom plane says 3 units of time went by, but the moving clock in the top plane has only ticked off 1 unit of time.

Similarly, observer in top plane says clocks in bottom plane run slow.

First step to a revised understanding of spacetime: find the replacement for the Galilean transformation relating an event's coords in 2 different IFs: (x', y', z', t') as a function of (x, y, z, t)

We get very far just from the assumption that both frames are inertial.

Freely moving clock in S: position $x_i(t)$; τ = time on clock

dx_i/dt = const (free particle) and $dt/d\tau$ = const (homogeneity)

=> $dx_\mu/d\tau$ = const $\qquad x_\mu(\mu = 1, 2, 3, 4) = (x, y, z, t)$

=> $d^2x_\mu/d\tau^2 = 0 \qquad$ Similarly, $d^2x'_\mu/d\tau^2 = 0$

$$\frac{dx'_\mu}{d\tau} = \sum_\nu \frac{\partial x'_\mu}{\partial x_\nu}\frac{dx_\nu}{d\tau}$$

$$\frac{d^2x'_\mu}{d\tau^2} = \sum_\nu \left[\frac{\partial x'_\mu}{\partial x_\nu}\cancelto{0}{\frac{d^2x_\nu}{d\tau^2}} + \sum_\sigma \frac{\partial^2 x'_\mu}{\partial x_\sigma \partial x_\nu}\frac{dx_\sigma}{d\tau}\frac{dx_\nu}{d\tau}\right] = 0 \quad \Rightarrow \quad \frac{\partial^2 x'_\mu}{\partial x_\sigma \partial x_\nu} = 0$$

$$\frac{\partial^2 x'_\mu}{\partial x_\sigma \partial x_\nu} = 0$$

=> the transformation is linear:

$$x' = A_1 x + B_1 y + C_1 z + D_1 t + E_1$$
$$y' = A_2 x + B_2 y + C_2 z + D_2 t + E_2$$
$$z' = A_3 x + B_3 y + C_3 z + D_3 t + E_3$$
$$t' = A_4 x + B_4 y + C_4 z + D_4 t + E_4$$

=> 20 parameters

Consider a stationary particle at the origin O' of S'. It must be moving at constant velocity in S. Pick axes in S such that O' is moving in pos direction along *x*-axis.

Pick origin of space and time in S such that O' is at O at $t = 0$
and origin of time in S' such that $t' = 0$ when O is at O' => $E_\mu = 0$

Also pick axes in S' such that free particle at O moves in neg direction along x'-axis.

v is the same for both origins ("reciprocity theorem"; end of sec 2.5 in Rindler)

O : $(x, y, z, t) = (0, 0, 0, t)$ and $(x', y', z', t') = (-vt', 0, 0, t')$

$y' = 0 = A_2\,x + B_2\,y + C_2\,z + D_2\,t = D_2\,t \quad \Rightarrow \quad D_2 = 0$

Similarly, $z' = 0 \quad \Rightarrow \quad D_3 = 0$

O' : $(x', y', z', t') = (0, 0, 0, t')$ and $(x, y, z, t) = (vt, 0, 0, t)$

$x' = 0 = A_1\,vt + B_1\,y + C_1\,z + D_1\,t \quad \Rightarrow \quad (A_1 v + D_1)\,t = 0 \quad \Rightarrow \quad D_1 = -A_1\,v$

$y' = 0 \quad \Rightarrow \quad D_2 = -A_2\,v \quad \Rightarrow \quad A_2 = 0$ (since we already found $D_2 = 0$)

$z' = 0 \quad \Rightarrow \quad D_3 = -A_3\,v \quad \Rightarrow \quad A_3 = 0$ (since we already found $D_3 = 0$)

Summary so far:

(11 parameters)

$$
\begin{aligned}
x' &= A_1\, x + B_1\, y + C_1\, z - A_1\, vt \\
y' &= B_2\, y + C_2\, z \\
z' &= B_3\, y + C_3\, z \\
t' &= A_4\, x + B_4\, y + C_4\, z + D_4\, t
\end{aligned}
$$

Isotropy of space plus symmetry between IFs
=> we can rotate coords in S such that $y' = y$ and $z' = z$

Since $y' = y$ and $z' = z$ AND x', t' can't depend on y', z'

$$B_1 = 0\,,\ C_1 = 0\,,\ B_4 = 0\,,\ \text{and}\ C_4 = 0$$

So:

$$
\begin{aligned}
x' &= A_1\,(x - vt) \\
y' &= y \\
z' &= z \\
t' &= A_4\, x + D_4\, t
\end{aligned}
$$

Down to 3 parameters.
We've only made use of def of IFs plus homogeneity and isotropy, to reduce to "standard configuration".

We have: $x' = A_1(x - vt)$ $\quad t' = A_4\, x + D_4\, t$

v – reversal: If we reverse the velocity, then S and S' exchange roles

$$\Rightarrow \; x = A_1(x' + vt')$$

Suppose, at this point, we assume, like Newton, an independent, universal time: $t' = t$ $\Rightarrow$ $A_1 vt = A_1 x - x' = x - A_1 x'$

$$\Rightarrow \; (A_1 - 1)\, x = -\, (A_1 - 1)\, x' \;\Rightarrow\; A_1 = 1$$

$$\Rightarrow \; x' = x - vt$$ (Galilean relativity)

Instead, use Einstein's 2nd postulate: suppose a light bulb flashes at O and O' when they are coincident.

Position of flash: $x = ct$ in S and $x' = ct'$ in S' $\Rightarrow$

$$ct = A_1(ct' + vt') = A_1 t'\,(c+v)$$
$$ct' = A_1(ct - vt) = A_1 t\,(c - v)$$

$\Rightarrow$

$$c^2 tt' = A_1^{\,2}\, tt'\,(c^2 - v^2)$$
$$A_1 = (1 - v^2/c^2\,)^{-1/2}$$

(pos root since $A_1 \to 1$ as $v \to 0$)

Notation: $\beta \equiv v/c \qquad \gamma \equiv (1 - \beta^2)^{-1/2}$

So: $x = \gamma\,(x' + vt')\quad ; \quad x' = \gamma\,(x - vt)$

$\Rightarrow\ \gamma\, x' = x - \gamma v t' = \gamma^2 x - \gamma^2 vt$

$\Rightarrow\ t' = \gamma\,(t - vx/c^2)$ Note: $1 - \gamma^2 = -\gamma^2 \beta^2$

Result:

$$x' = \gamma\,(x - vt)$$
$$y' = y$$
$$z' = z$$
$$t' = \gamma\,(t - vx/c^2)$$

"standard Lorentz transformation"

To demonstrate full consistency with 2nd postulate, we must show that light propagates with speed c in all directions, not just along x and x' axes.

LT yields: $c^2 dt'^2 - dr'^2 = c^2 dt^2 - dr^2 \quad ; \quad dr^2 = dx^2 + dy^2 + dz^2$

Light front propagating in Euclidean space has $dr^2 = c^2 dt^2$

So, light propagating with speed c in any dir in S => same thing in S'

Properties and Consequences of the LT

$x' = \gamma\,(x - vt)$ $\quad y' = y$ $\quad z' = z$ $\quad t' = \gamma\,(t - vx/c^2)$

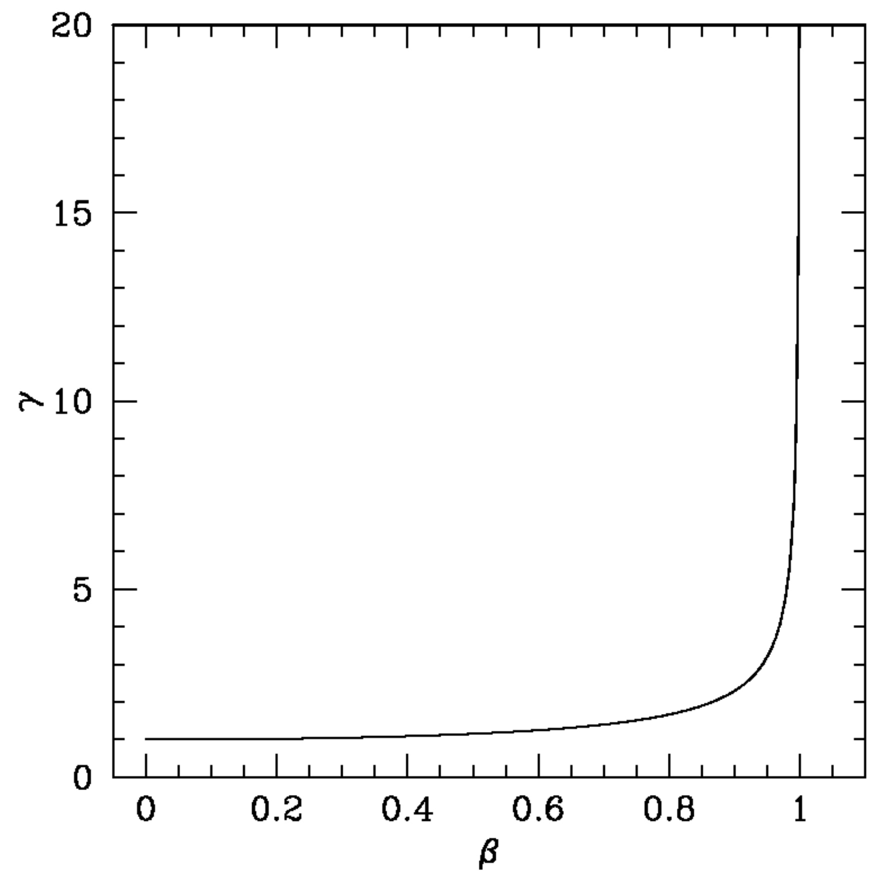
1. Relativity of simultaneity: $t' \neq t$

If you have a lattice of stationary, synchronized clocks in S, they are not synchronized in S'.

2. Symmetry in x and ct :

$$ct' = \gamma\,(ct - \beta x) \quad ; \quad x' = \gamma\,(x - \beta ct)$$

3. Lorentz factor γ :



4. Newtonian limit:

Lorentz transformation →
Galilean trans. when $\beta \to 0$

β has to be substantial before Newtonian limit is noticeably violated => SR is not intuitive

5. Since LT is linear, coord differences transform in the same way as the coords themselves:

$$\Delta x' = \gamma\,(\Delta x - v\Delta t) \qquad \Delta y' = \Delta y \qquad \Delta z' = \Delta z \qquad \Delta t' = \gamma\,(\Delta t - v\Delta x/c^2)$$

$$dx' = \gamma\,(dx - v\,dt) \qquad dy' = dy \qquad dz' = dz \qquad dt' = \gamma\,(dt - v\,dx/c^2)$$

6. Squared displacement (between 2 events):

$\mathbf{\Delta s}^2 \equiv c^2\,\Delta t^2 - \Delta x^2 - \Delta y^2 - \Delta z^2$ is a scalar: has the same value in all IFs.

$\Delta \mathbf{s}^2$ can be pos, neg, or 0.

$\Delta s \equiv |\Delta \mathbf{s}^2|^{1/2}$ is called the "interval" between 2 events

7. Length contraction: rod at rest in S', lying along x'-axis (= x-axis)

Length = distance btwn the 2 endpoints, observed **simultaneously**

Observations need not be simultaneous in S', the rod's "rest frame".

$\Delta x' = \gamma\,(\Delta x - v\Delta t)$

For observer in S, $\Delta t = 0$ => $\Delta x' = \gamma \Delta x$

=> $L = L_0/\gamma$; L_0 is called the "rest length" or "proper length".

Barn and ladder paradox (Rindler sec 3.4)

A farmer wants to fit a 20-foot ladder into a 10-foot barn, so he runs with the ladder towards the barn at $v = 0.866\,c$ (γ=2). His buddy closes the door once the ladder is in. As the ladder comes to rest in the barn's frame, it will tend to assume, if it can, its original length relative to the barn. So, it must shatter, bend, burst through the door, or remain compressed.

But how does this play out in the frame of the moving ladder?

Ladder is 20 feet long and barn is 5 feet long. Both inertial observers must find the same physical result (e.g., busted ladder in closed barn). How does this happen in the frame of the moving ladder?

There is no direct experimental verification of length contraction, but there is indirect evidence (involving current in a wire; more later).

8. Time dilation: clock at rest in S' ticks off time $\Delta t'$; delineated by 2 events separated by $\Delta t'$ and $\Delta x' = 0$.

How much time has elapsed in S? $\Delta t = \gamma\,(\Delta t' + v\,\Delta x'/c^2)$

=> $\Delta t = \gamma\,\Delta t'$ or $T = \gamma T_0$

Clock moving rel to S runs slow compared with clock at rest in S.

Time dilation has been experimentally confirmed in several ways. For example: flying atomic clocks in airplanes. Muons formed when cosmic rays strike the top of the atmosphere reach ground in much larger numbers than expected based on half-life.

The Twin Paradox: one twin has to accelerate.

Length contraction and time dilation are due to the laws of physics, which can be quite complicated (e.g., cohesive forces between atoms in a solid). We may not know the details of all this physics, but we know that the length contraction and time dilation formulas must be satisfied. These effects are not some illusion or error of measurement.

SR is a meta-theory.

9. Velocity transformation: particle moving with velocity $\mathbf{u} = (u_x, u_y, u_z)$ rel to S and $\mathbf{u}' = (u'_x, u'_y, u'_z)$ rel to S'

$u_x = dx/dt \qquad u_y = dy/dt \qquad u_z = dz/dt$

$$u'_x = \frac{dx'}{dt'} = \frac{\gamma\left(dx - vdt\right)}{\gamma\left(dt - vdx/c^2\right)} = \frac{u_x - v}{1 - vu_x/c^2}$$

$$u'_y = \frac{dy'}{dt'} = \frac{dy}{\gamma\left(dt - vdx/c^2\right)} = \frac{u_y}{\gamma\left(1 - vu_x/c^2\right)}$$

$$u'_z = \frac{u_z}{\gamma\left(1 - vu_x/c^2\right)}$$

v-reversal yields "velocity addition formulae":

$$u_x = \frac{u'_x + v}{1 + vu'_x/c^2} \qquad u_y = \frac{u'_y}{\gamma\left(1 + vu'_x/c^2\right)} \qquad u_z = \frac{u'_z}{\gamma\left(1 + vu'_x/c^2\right)}$$

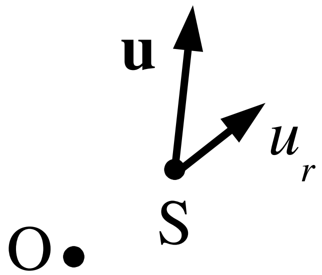
(gives resultant, **u**, of first giving particle velocity **v** = (*v*, 0, 0) and then, rel to its new rest frame, another velocity **u'**)

A little algebra yields: $$c^2 - u^2 = \frac{c^2(c^2 - u'^2)(c^2 - v^2)}{(c^2 + vu'_x)^2}$$

=> If particle starts with $v < c$, it can never reach c.

(acceleration during an infinitesimal time step yields an infinitesimal u' => all the terms in above eqn are positive)

10. The Doppler effect: Suppose a light source is moving with speed u relative to observer, with radial component u_r



dt_0 = time interval btwn crests in source's frame

time dilation: in observer's frame, pulses occur at interval $dt = \gamma dt_0$

During dt, source moves a distance $u_r\, dt = u_r\, \gamma dt_0$ away from observer.

=> extra time interval btwn crests of $u_r\, \gamma\, dt_0/c$

=> $dt = \gamma\, dt_0\, (1+u_r/c)$

frequency $\nu \propto dt^{-1}$ =>
$$\frac{\nu_0}{\nu} = \frac{dt}{dt_0} = \gamma(1 + u_r/c) = \frac{1 + u_r/c}{(1 - u^2/c^2)^{1/2}}$$

if $u/c << 1$:
$$\approx \left(1 + \frac{u_r}{c}\right)\left(1 + \frac{1}{2}\frac{u^2}{c^2}\right)$$
$$= 1 + \frac{u_r}{c} + \frac{1}{2}\frac{u^2}{c^2} + O(u^3/c^3)$$

If motion is purely radial:

$$\frac{\nu_0}{\nu} = \frac{1 + u/c}{(1 - u^2/c^2)^{1/2}} = \left(\frac{1 + u/c}{1 - u/c}\right)^{1/2}$$

$\boxed{\text{SP 2.1-6}}$