Spacetime and 4-vectors

Minkowski space = 4-dimensional spacetime ≠ Euclidean 4-space

Each point in Minkowski space is an event.

In SR, Minkowski space is an absolute structure (like space in Newtonian theory).

Euclidean 3-space: each point has coords (x, y, z); the "metric" is the distance btwn 2 points: $dr^2 = dx^2 + dy^2 + dz^2$

dr^2 is invariant under rotation of coords

Minkowski space: each point has coords (x, y, z, ct); the metric is

$$ds^2 = c^2dt^2 - dx^2 - dy^2 - dz^2$$

ds^2 is invariant under LTs; like rotation of coords in Minkowski space!

Euclidean space: translations and rotations of coord axes => infinite
number of different Cartesian coord systems

Minkowski space: translations and rotations of coord axes plus LTs
=> infinite number of different coord systems
(different IFs with Cartesian coord systems)

Review of vectors in Euclidean 3-space ("3-vectors")

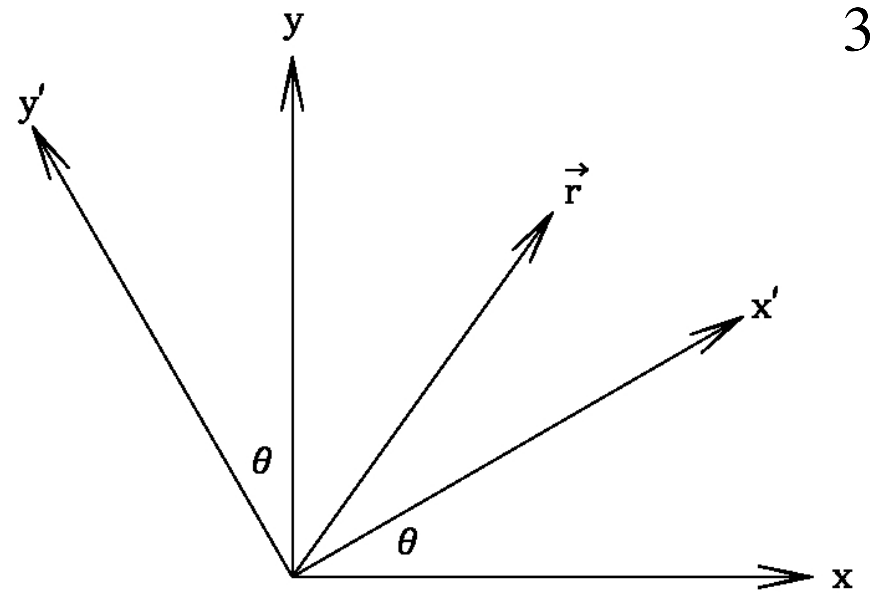
Vectors are absolute quantities, independent of coord system.

Representation (x, y, z) depends on coord system chosen.

Prototypical vector: the displacement $\Delta\mathbf{r}$

Definition of a 3-vector: object with 3 components that transforms
under a change of coord system (rotations and translations)
in exactly the same way as the displacement.

Coordinate transformations are accomplished using matrix algebra.



e.g.: rotation of coord axes about z-axis:

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

$$x' = \cos\theta \; x + \sin\theta \; y$$

$$y' = -\sin\theta \; x + \cos\theta \; y$$

$$z' = z$$

$$\mathbf{r}' = \mathbf{R} \cdot \mathbf{r}$$

Note that the transformation eqns are linear.

translation: $\mathbf{r}' = \mathbf{r} + \mathbf{r}_0$

Scalar: a number that doesn't depend on coord system

In Newtonian theory, time and mass are scalars.

Equations involving only vectors and scalars are invariant; i.e, if true in one coord system, then true in all coord systems.

Laws of physics must have this property.

Rotations and translations are linear transformations
=> sum of 2 vectors is a vector
scalar times vector is a vector

Derivative of vector wrt scalar is a vector.

Magnitude $a = |\mathbf{a}|$ of 3-vector $\mathbf{a} = (a_1, a_2, a_3)$: $a^2 = a_1{}^2 + a_2{}^2 + a_3{}^2$

Magnitude is a scalar, since $\mathbf{a}$ transforms like $\mathbf{\Delta r}$ and $\mathbf{\Delta r}^2$ is invariant.

Scalar product: $\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2 + a_3 b_3$

It's a scalar since $|\mathbf{a} + \mathbf{b}|^2 = (a_1 + b_1)^2 + (a_2 + b_2)^2 + (a_3 + b_3)^2$

$$= a^2 + b^2 + 2\,(a_1 b_1 + a_2 b_2 + a_3 b_3)$$

Vectors in Minkowski space (4-vectors)

Notation: 3-vectors will be lowercase (e.g., **r**); 4-vectors uppercase (**R**)

Prototypical 4-vector: displacement $\Delta\mathbf{R} = (\Delta x, \Delta y, \Delta z, \Delta ct)$

Definition of 4-vector: an object with 4 components that transforms like $\Delta\mathbf{R}$ under a Poincaré transformation (includes standard LTs, rotations, translations, and successive applications of these)

Matrix representation of standard LT:

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} \gamma & 0 & 0 & -\gamma\beta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\gamma\beta & 0 & 0 & \gamma \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}$$

4-vectors are absolute entities; components differ in different IFs.

An equation involving only 4-vectors and scalars that is true in one IF is true in all IFs; called Lorentz-invariant, or covariant, equations.

SR: Laws of physics must be Lorentz-invariant. (SR as meta-theory)

Square of a 4-vector $\mathbf{A} = (A_1, A_2, A_3, A_4)$: $\mathbf{A}^2 = A_4{}^2 - A_1{}^2 - A_2{}^2 - A_3{}^2$

$\mathbf{A}^2$ can be pos, neg, or 0 and is a scalar, since $\Delta\mathbf{R}^2$ is invariant.

Magnitude $A = |\mathbf{A}^2|^{1/2}$ is also obviously a scalar.

=> scalar product $\mathbf{A}\cdot\mathbf{B} = A_4B_4 - A_1B_1 - A_2B_2 - A_3B_3$ is a scalar.

Interval of proper time: $d\tau^2 \equiv \boldsymbol{ds}^2/c^2 = dt^2 - (dx^2 + dy^2 + dz^2)/c^2$

$d\tau$ is a scalar and corresponds to time ticked on a clock at rest.

For a particle with 3-velocity $\mathbf{u}$, $dt = \gamma(u)\, d\tau$

4-velocity:

$$\mathbf{U} \equiv \frac{d\mathbf{R}}{d\tau} = \left(\frac{dx}{d\tau}, \frac{dy}{d\tau}, \frac{dz}{d\tau}, \frac{d(ct)}{d\tau}\right) = \gamma(u)(\mathbf{u}, c)$$

4-acceleration:

$$\mathbf{A} = \frac{d\mathbf{U}}{d\tau} = \frac{d\mathbf{U}}{dt}\frac{dt}{d\tau} = \gamma\frac{d\mathbf{U}}{dt}$$

$$= \gamma\left(\frac{d(\gamma\mathbf{u})}{dt}, \frac{d(\gamma c)}{dt}\right) = \gamma\left(\frac{d\gamma}{dt}\mathbf{u} + \gamma\mathbf{a}, c\frac{d\gamma}{dt}\right)$$

$\mathbf{U} = \gamma(\mathbf{u}, c)$ $\mathbf{A} = \gamma(\gamma' \mathbf{u} + \gamma \mathbf{a}, \gamma' c)$ $(\gamma' = d\gamma/dt)$

In instantaneous rest frame ($\mathbf{u} = 0$):

$$\frac{d\gamma}{dt} = \frac{d(1 - u^2/c^2)^{-1/2}}{dt} = \left(1 - \frac{u^2}{c^2}\right)^{-3/2} \frac{u}{c^2}\frac{du}{dt} = 0 \quad \Rightarrow \quad \mathbf{A} = (\mathbf{a}, 0)$$

Square of 4-velocity, evaluated in arbitrary frame:

$$\mathbf{U}^2 = \gamma^2(c^2 - u^2) = c^2\frac{1 - u^2/c^2}{1 - u^2/c^2} = c^2$$

Evaluating in rest frame yields $\mathbf{U}^2 = c^2$ immediately.

This is a trivial example of the power of relativity: evaluate scalars in the frame that makes the calculation the simplest.

Square of 4-accel: $\mathbf{A}^2 = -\alpha^2$ (α = mag of 3-accel in rest frame "proper acceleration")

SP 3.1-3

Rapidity θ is defined by: $\cosh\theta = \gamma$; $\sinh\theta = \gamma\beta$ => $\tanh\theta = \beta$

Recall: $\cosh\theta = (e^{\theta} + e^{-\theta})/2$; $\sinh\theta = (e^{\theta} - e^{-\theta})/2$

$\cosh^2\theta - \sinh^2\theta = 1$ => $\gamma^2 - \gamma^2\beta^2 = 1$ => definition is consistent.

Standard LT: $x' = \gamma(x - \beta ct) = \cosh\theta\, x - \sinh\theta\, ct$

$ct' = \gamma(ct - \beta x) = -\sinh\theta\, x + \cosh\theta\, ct$

$$\begin{pmatrix} x' \\ y' \\ z' \\ ct' \end{pmatrix} = \begin{pmatrix} \cosh\theta & 0 & 0 & -\sinh\theta \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh\theta & 0 & 0 & \cosh\theta \end{pmatrix} \begin{pmatrix} x \\ y \\ z \\ ct \end{pmatrix}$$

Note similarity to rotation matrix.

Consider 2 boosts along *x*-axis: β_1 followed by β_2, yielding β_3

vel add formula:

$$u_x = \frac{u'_x + v}{1 + v u'_x / c^2}$$

$$\beta_3 = \frac{\beta_2 + \beta_1}{1 + \beta_1 \beta_2}$$

$$\tanh\theta_3 = \frac{\tanh\theta_1 + \tanh\theta_2}{1 + \tanh\theta_1 \tanh\theta_2} = \tanh(\theta_1 + \theta_2)$$

$\theta_3 = \theta_1 + \theta_2$; rapidity, unlike velocity, adds!

SP 3.4

Nomenclature: A 4-vector **A** is ...

timelike	if	$\mathbf{A}^2 > 0$
lightlike, or null,	if	$\mathbf{A}^2 = 0$
spacelike	if	$\mathbf{A}^2 < 0$

in analogy with prototype $\Delta\mathbf{s}^2 = c^2\Delta t^2 - \Delta\mathbf{r}^2$

$\Delta\mathbf{s}^2 = 0$ => light can travel from earlier to later event

$\Delta\mathbf{s}^2 > 0$ => particle moving with $v < c$ can travel from earlier to later event; can transform to a frame where $\Delta\mathbf{r}^2 = 0$

$\Delta\mathbf{s}^2 < 0$ => no particle can travel from one event to the other with $v \leq c$; can transform to a frame where $\Delta t = 0$

Null and timelike 4-vectors are classed together as causal vectors:
sign of 4th component is invariant ("future-pointing" if $A_4 > 0$
"past-pointing" if $A_4 < 0$)