Relativistic Mechanics

SR as a meta-theory: laws of physics must be Lorentz-invariant.

Laws of classical mechanics are not: they are invariant under Galilean transformations, not Lorentz transformations.

=> we need a revised, relativistic, mechanics!

Starting point: Assume each particle has an invariant proper mass (or rest mass) m_0

Define 4-momentum $\mathbf{P} = m_0\,\mathbf{U}$

Basic axiom: conservation of **P** in particle collisions: $\Sigma^* \mathbf{P}_n = 0$

(in the sum, **P** of each particle going into the collision is counted positively while **P** of each particle coming out is counted negatively)

Since a sum of 4-vectors is a 4-vector, this equation is Lorentz-invariant.

$\mathbf{P} = m_0\, \mathbf{U} = m_0\, \gamma(u)\, (\mathbf{u}, c) \equiv (\mathbf{p}, mc)$

relativistic mass: $m = \gamma(u)\, m_0$

relativistic momentum: $\mathbf{p} = m\, \mathbf{u}$

Conservation of 4-momentum yields both: $\Sigma^* \, \mathbf{p} = 0$ $\Sigma^* \, m = 0$

These are Newtonian conservation laws when $c \rightarrow \infty$

What about conservation of energy?

When $v/c \ll 1$: $m = m_0(1 - v^2/c^2)^{-1/2} \approx m_0 \left(1 + \frac{1}{2}\frac{v^2}{c^2}\right)$

$$mc^2 \approx m_0 c^2 + \frac{1}{2} m_0 v^2$$

Einstein proposed the equivalence of mass and energy: $E = mc^2$

=> $\mathbf{P} = (\mathbf{p}, E\,/\,c)$

Relativistic energies

rest energy: $m_0 c^2$

kinetic energy: $T = mc^2 - m_0 c^2 = (\gamma - 1)\, m_0 c^2$

Note: Potential energy associated with position of particle in an external electromagnetic or gravitational field does not contribute to the relativistic mass of a particle. When particle's PE changes, it's actually the energy of the field that's changing.

H atom: energy of electron = -13.6 eV (binding energy)

$$\Rightarrow \; m_{\text{atom}}\, c^2 = m_p\, c^2 + m_e\, c^2 - 13.6 \text{ eV}$$

=> atom has less mass than its constituents!

$1 \text{ eV} = 1.602 \times 10^{-12} \text{ ergs} = 1.602 \times 10^{-19} \text{ J}$

$m_p = 1.67 \times 10^{-24}$ g ; $m_e = 9.11 \times 10^{-28}$ g

$$\Rightarrow \; \Delta m / m = 13.6 \text{ eV} / (m_p + m_e)\, c^2 \approx 1.8 \times 10^{-8}$$

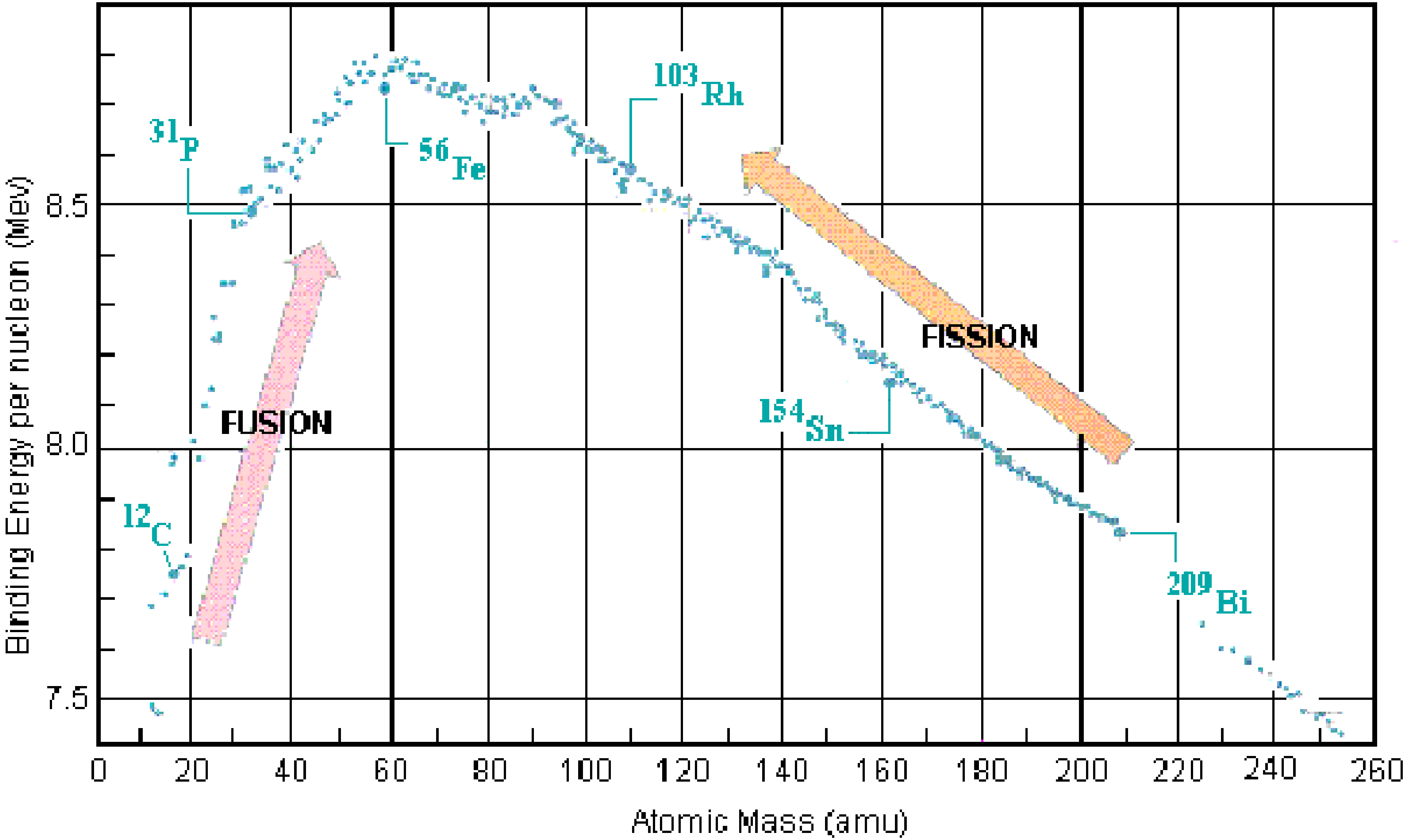
Nuclear physicists use the atomic mass unit u, defined so that the mass of the ^{12}C atom is exactly 12 u.

$1\ \text{u} = 931.502\ \text{MeV} / c^2$

$m_p = 1.00727647\ \text{u}$; $m_n = 1.00866501\ \text{u}$; $m_e = 5.485803 \times 10^{-4}\ \text{u}$

=> $\Delta m / m \approx 0.8\%$ for the ^{12}C nucleus

Binding energy ≈ 90 MeV (≈ 7.5 MeV per nucleon)



Binding Energy per nucleon (Mev)
8.5
8.0
7.5
0
20
40
60
80
100
120
140
160
180
200
220
240
260
Atomic Mass (amu)
FUSION
FISSION
31P
56Fe
103Rh
12C
154Sn
209Bi

$$\mathbf{P} = m_0\,\gamma(u)\,(\mathbf{u}, c) = (\mathbf{p}, mc) = (\mathbf{p}, E/c)$$

$$\Rightarrow \mathbf{P}^2 = E^2/c^2 - \mathbf{p}^2$$

In particle's rest frame: $\mathbf{P} = (\mathbf{0}, m_0 c) \Rightarrow \mathbf{P}^2 = m_0^{\,2}\, c^2$

$$\Rightarrow E^2/c^2 - \mathbf{p}^2 = m_0^{\,2}\, c^2 \quad \Rightarrow \quad E^2 = m_0^{\,2}\, c^4 + \mathbf{p}^2 c^2$$

For a photon: $m_0 = 0 \quad \Rightarrow \quad E = pc \quad , \quad \mathbf{P} = E/c\,(\mathbf{n}, 1)$

$E = h\nu = hc/\lambda \quad , \quad \mathbf{n}$ = a unit vector in dir of propagation

Since **P** is a 4-vector, *E* and **p** transform as follows under a standard LT:

$$p_x' = \gamma(p_x - \beta E/c) \qquad E' = \gamma(E - \beta p_x c)$$

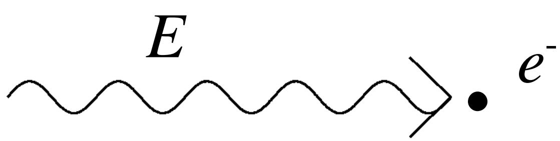
$$p_y' = p_y$$

$$p_z' = p_z$$

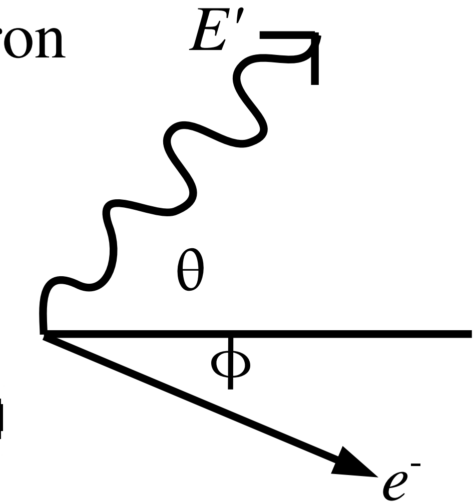
Applications

1. Compton scattering: photon strikes stationary electron

Before:



After:



$$\mathbf{P}_{e,i} = m_e c\,(\mathbf{0}, 1) \qquad \mathbf{P}_{\gamma,f} = \frac{E'}{c}(\hat{x}\cos\theta + \hat{y}\sin\theta, 1)$$

$$\mathbf{P}_{\gamma,i} = \frac{E}{c}(\hat{x}, 1)$$

$$\mathbf{P}_{e,i} + \mathbf{P}_{\gamma,i} = \mathbf{P}_{e,f} + \mathbf{P}_{\gamma,f}$$

$$(\mathbf{P}_{e,i} + \mathbf{P}_{\gamma,i} - \mathbf{P}_{\gamma,f})^2 = \mathbf{P}_{e,f}^2 = m_e^2 c^2$$

$$m_e^2 c^2 + (\mathbf{P}_{\gamma,i} - \mathbf{P}_{\gamma,f})^2 + 2\mathbf{P}_{e,i}\cdot(\mathbf{P}_{\gamma,i} - \mathbf{P}_{\gamma,f}) = m_e^2 c^2$$

$$-2\mathbf{P}_{\gamma,i}\cdot\mathbf{P}_{\gamma,f} + 2\mathbf{P}_{e,i}\cdot\mathbf{P}_{\gamma,i} - 2\mathbf{P}_{e,i}\cdot\mathbf{P}_{\gamma,f} = 0$$

$$\mathbf{P}_{e,i} = m_e c\,(\mathbf{0}, 1)$$

$$\mathbf{P}_{\gamma,f} = \frac{E'}{c}\,(\hat{x}\cos\theta + \hat{y}\sin\theta, 1)$$

$$\mathbf{P}_{\gamma,i} = \frac{E}{c}\,(\hat{x}, 1)$$

$$\mathbf{P}_{e,i}\cdot\mathbf{P}_{\gamma,i} = \mathbf{P}_{e,i}\cdot\mathbf{P}_{\gamma,f} + \mathbf{P}_{\gamma,i}\cdot\mathbf{P}_{\gamma,f}$$

$$m_e E = m_e E' + \frac{EE'}{c^2}(1-\cos\theta)$$

$$\frac{hc}{\lambda} = \frac{hc}{\lambda'} + \frac{h^2c^2\,(1-\cos\theta)}{m_e c^2 \lambda\lambda'}$$

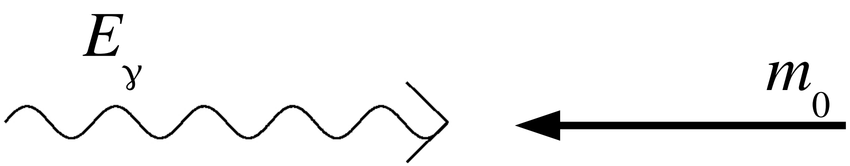
$$\lambda' = \lambda + \frac{h}{m_e c}(1-\cos\theta)$$

$h / m_e c$ is called the "Compton wavelength" of the electron.

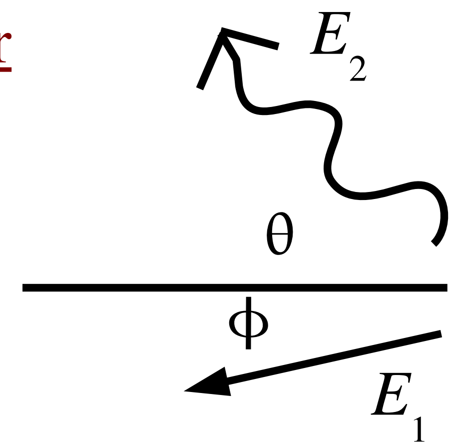
SP 4.1

2. Inverse Compton scattering: photon scatters off a highly energetic charged particle

Before



After



$$\mathbf{P}_{p,i} = (-p\hat{x}, E/c) \qquad \mathbf{P}_{\gamma,i} = \frac{E_\gamma}{c}(\hat{x}, 1)$$

$$\mathbf{P}_{\gamma,f} = \frac{E_2}{c}(-\hat{x}\cos\theta + \hat{y}\sin\theta, 1)$$

As before: $\mathbf{P}_{p,i} \cdot \mathbf{P}_{\gamma,i} = \mathbf{P}_{p,i} \cdot \mathbf{P}_{\gamma,f} + \mathbf{P}_{\gamma,i} \cdot \mathbf{P}_{\gamma,f}$

=> $E_\gamma(E + pc) = E_2(E - pc\cos\theta) + E_\gamma E_2(1 + \cos\theta)$

Assume $E \gg m_0 c^2 \gg E_\gamma$

$$pc = (E^2 - m_0^2c^4)^{1/2} \approx E\left(1 - \frac{m_0^2c^4}{2E^2}\right) \qquad (E \gg m_0c^2)$$

$$E_\gamma(E + pc) = E_2(E - pc\cos\theta) + E_\gamma E_2(1 + \cos\theta)$$

(from previous slide)

$$2EE_\gamma \approx E_2\left(E - E\cos\theta + \frac{m_0^2c^4}{2E}\cos\theta + E_\gamma + E_\gamma\cos\theta\right)$$

$$= E_2\left[(E + E_\gamma) - \left(E - E_\gamma - \frac{m_0^2c^4}{2E}\right)\cos\theta\right]$$

Maximum energy is transferred to photon when cos θ = 1:

$$2EE_\gamma \approx 2E_2E_\gamma\left(1 + \frac{m_0^2c^4}{4EE_\gamma}\right) \quad \Rightarrow \quad E_2 = \frac{E}{1 + m_0^2c^4/4EE_\gamma}$$

Cosmic Microwave Background (CMB): 3 K blackbody spectrum

=> typical photon energy ~ $kT = 3 \times 10^{-4}$ eV

Very high energy cosmic ray proton: 10^{20} eV ; $m_p c^2 = 938$ MeV

=> CMB photon can be upscattered to ~ 10^{19} eV (source of gamma rays)

3. Particle accelerators: "available energy" for creating new particles is the energy in the CM frame

Consider 2 arrangments:

a) a 30 GeV proton collides with a proton at rest

b) two 15 GeV protons collide head-on

(lab and CM frames are identical => 30 GeV of available energy)

a) lab frame: $\mathbf{P}_i = [\ \gamma\beta m_p c, (\gamma+1)\ m_p c\ ]$ CM frame: $\mathbf{P} = (\mathbf{0}, E/c)$

$$\mathbf{P}^2 = (\gamma+1)^2 m_p^2 c^2 - \gamma^2\beta^2 m_p^2 c^2 = E^2/c^2$$

$$[(\gamma^2 - \gamma^2\beta^2) + 1 + 2\gamma] m_p^2 c^2 = E^2/c^2$$

$$\begin{aligned} E^2 &= 2(1+\gamma) m_p^2 c^4 \\ &= 2 m_p c^2 [m_p c^2 + \gamma m_p c^2] \\ &= 2(0.938\,\mathrm{GeV})[0.938\,\mathrm{GeV} + 30\,\mathrm{GeV}] \end{aligned}$$

$$\Rightarrow E = 7.6\,\mathrm{GeV}$$

To reach an available energy of 30 GeV in arrangement (a), we would need a 480 GeV proton!

4. Cosmic ray cutoff: high-energy nucleons can undergo the reaction $\gamma + N \rightarrow N + \pi$

Reactions with CMB photons ($E \sim 3 \times 10^{-4}$ eV) probably produces a cutoff in the cosmic ray spectrum: no CRs with energy above the threshold energy for the reaction.

Assuming a head-on collision, what's the threshold energy?

We need $E = (m_N + m_\pi)\, c^2$ in CM frame => $\mathbf{P}_f = (\mathbf{0}, m_N c + m_\pi c)$

Lab frame, before collision: $\mathbf{P}_\gamma = \frac{E_\gamma}{c}(\hat{x}, 1)$ $\quad\quad \mathbf{P}_\mathrm{N} = (-p_\mathrm{N}\hat{x}, E_\mathrm{N}/c)$

$$\Rightarrow \mathbf{P}_i = \left(\frac{E_\gamma}{c} - p_\mathrm{N}, 0, 0, \frac{E_\gamma + E_\mathrm{N}}{c}\right)$$

$$(E_\gamma + E_\mathrm{N})^2 - (E_\gamma - p_\mathrm{N}c)^2 = (m_\mathrm{N} + m_\pi)^2 c^4$$

$$(E_\gamma + E_\mathrm{N})^2 - (E_\gamma - p_\mathrm{N}c)^2 = (m_\mathrm{N} + m_\pi)^2 c^4$$

$$E_\mathrm{N}^2 + 2E_\gamma E_\mathrm{N} - p_\mathrm{N}^2 c^2 + 2E_\gamma p_\mathrm{N} c = (m_\mathrm{N} + m_\pi)^2 c^4$$

$$2E_\gamma(E_\mathrm{N} + p_\mathrm{N}c) = [(m_\mathrm{N} + m_\pi)^2 - m_\mathrm{N}^2]c^4$$

$$E_\mathrm{N} + p_\mathrm{N}c = \frac{m_\pi(2m_\mathrm{N} + m_\pi)c^4}{2E_\gamma} \approx 6 \times 10^{14}\ \mathrm{MeV}$$

$$(m_\mathrm{N}c^2 \approx 940\ \mathrm{MeV}\ ,\ \ m_\pi c^2 \approx 140\ \mathrm{MeV})$$

=> $E_\mathrm{N} \gg m_\mathrm{N}c^2$ => LHS $\rightarrow 2\,E_\mathrm{N}$ => $E_\mathrm{N} \approx 3 \times 10^{14}$ MeV

SP 4.2-8

What about forces?

Define 4-force $\mathbf{F} = d\mathbf{P}/d\tau = d(m_0\mathbf{U}) / d\tau = m_0\mathbf{A} + (dm_0/d\tau)\,\mathbf{U}$

Limited version of Newton's 3rd Law for contact collisions between 2 particles: during contact, τ is the same for both particles.

$\mathbf{F}_1 + \mathbf{F}_2 = d(\mathbf{P}_1 + \mathbf{P}_2) / d\tau = 0 \quad \Rightarrow \quad \mathbf{F}_2 = - \mathbf{F}_1$

Relativistic 3-force $\mathbf{f} = d\mathbf{p}/dt = d(m\mathbf{u})/dt$

$$\mathbf{F} = \frac{\mathbf{dP}}{d\tau} = \gamma(u)\frac{d}{dt}(\mathbf{p}, E/c) = \gamma(u)\left(\mathbf{f}, \frac{1}{c}\frac{dE}{dt}\right)$$

f becomes Newtonian force when $\beta \to 0$

4th component of **F** is proportional to the power absorbed by the particle.

$$\mathbf{F}\cdot\mathbf{U} = \left(m_0\mathbf{A} + \frac{dm_0}{d\tau}\mathbf{U}\right)\cdot\mathbf{U} = c^2\frac{dm_0}{d\tau}$$

=> $\mathbf{F}\cdot\mathbf{U}$ = proper rate at which particle's internal energy increases

Also:

$$\begin{aligned}\mathbf{F}\cdot\mathbf{U} &= \gamma(u)\left(\mathbf{f}, \frac{1}{c}\frac{dE}{dt}\right)\cdot\gamma(u)(\mathbf{u}, c)\\ &= \gamma^2(u)\left(\frac{dE}{dt} - \mathbf{f}\cdot\mathbf{u}\right)\end{aligned}$$

Equating above 2 expressions for $\mathbf{F}\cdot\mathbf{U}$ => rest mass-preserving forces are characterized by:

$\mathbf{F}\cdot\mathbf{U} = 0$

$\mathbf{f}\cdot\mathbf{u} = dE/dt$ => $\mathbf{f}\cdot d\mathbf{r} = dE$

$\mathbf{F} = \gamma(u)\,(\mathbf{f}, \mathbf{f}\cdot\mathbf{u}/c)$

(as in Newtonian theory; the added energy is entirely kinetic)

4-force transformation: $(Q \equiv dE/dt)$

$$f_1' = \frac{f_1 - vQ/c^2}{1 - u_1 v/c^2} \quad \left(= \frac{f_1 - v\mathbf{f}\cdot\mathbf{u}/c^2}{1 - u_1 v/c^2}\,, \quad \mathrm{m}_0 = \mathrm{const} \right)$$

$$f_2' = \frac{f_2}{\gamma(v)(1 - u_1 v/c^2)}$$

$$f_3' = \frac{f_3}{\gamma(v)(1 - u_1 v/c^2)}$$

$$Q' = \frac{Q - v f_1}{1 - u_1 v/c^2}$$

Note: for rest mass-preserving force, **f** is invariant among IFs with relative velocities parallel to force.

SP 4.9

For a rest mass-preserving force:

$$\mathbf{f} = \frac{d(m\mathbf{u})}{dt} = m\mathbf{a} + \frac{dm}{dt}\mathbf{u}$$
$$= \gamma m_0 \mathbf{a} + \frac{1}{c^2}\frac{dE}{dt}\mathbf{u}$$
$$= \gamma m_0 \mathbf{a} + \frac{\mathbf{f}\cdot\mathbf{u}}{c^2}\mathbf{u}$$

a is not generally parallel to **f**

$$\mathbf{f} \parallel \mathbf{u}: \quad f_{\|} = \gamma m_0 a_{\|} + f_{\|} u^2/c^2$$

$$\Rightarrow f_{\|} = \gamma^3 m_0 a_{\|}$$

$$\mathbf{f} \perp \mathbf{u}: \quad f_{\perp} = \gamma m_0 a_{\perp}$$

A moving particle offers different inertial resistances to the same force, depending on whether it is applied longitudinally or transverse.

Also for a rest mass-preserving force:

$$\mathbf{F} = m_0\mathbf{A} + dm_0/d\tau\ \mathbf{U} = m_0\, d^2\mathbf{R}/d\tau^2 = \gamma(u)\,(\mathbf{f},\, c^{-1}\, dE/dt)$$

$$\Rightarrow \quad \gamma(u)\,\mathbf{f} = m_0\, d^2\mathbf{r}/d\tau^2$$