Introduction to Tensors

Contravariant and covariant vectors

Rotation in 2-space: $x' = \cos\theta\, x + \sin\theta\, y$

$y' = -\sin\theta\, x + \cos\theta\, y$

To facilitate generalization, replace (x, y) with (x^1, x^2)

Prototype contravariant vector: $d\mathbf{r} = (dx^1, dx^2)$

$$dx^{1'} = \frac{\partial x^{1'}}{\partial x^1}\, dx^1 + \frac{\partial x^{1'}}{\partial x^2}\, dx^2 \qquad = \cos\theta\, dx^1 + \sin\theta\, dx^2$$

Similarly for $dx^{2'}$

Same holds for $\Delta\mathbf{r}$, since transformation is linear.

Compact notation: $dx^{i'} = \sum_j \frac{\partial x^{i'}}{\partial x^j} dx^j$

(generalizes to any transformation in a space of any dimension)

Contravariant vector: $a^{i'} = \sum_j \frac{\partial x^{i'}}{\partial x^j} a^j$

Now consider a scalar field $\phi(\mathbf{r})$: How does $\nabla\phi$ transform under rotations?

$$\nabla\phi = \left(\frac{\partial\phi}{\partial x^1}, \frac{\partial\phi}{\partial x^2}\right)$$

$$\frac{\partial\phi}{\partial x^{i'}} = \sum_j \frac{\partial\phi}{\partial x^j} \frac{\partial x^j}{\partial x^{i'}}$$

$$\nabla'\phi = \left(\frac{\partial\phi}{\partial x^{1'}}, \frac{\partial\phi}{\partial x^{2'}}\right)$$

$\frac{\partial x^j}{\partial x^{i'}}$ appears rather than $\frac{\partial x^{i'}}{\partial x^j}$

For rotations in Euclidean n-space:

$$\frac{\partial x^j}{\partial x^{i'}} = \frac{\partial x^{i'}}{\partial x^j} = \cos\theta$$ where θ = angle btwn x^j and $x^{i'}$ axes

It is not the case for all spaces and transformations that $\frac{\partial x^j}{\partial x^{i'}} = \frac{\partial x^{i'}}{\partial x^j}$

so we define a new type of vector that transforms like the gradient:

Covariant vectors: $a_{i'} = \sum_j a_j \frac{\partial x^j}{\partial x^{i'}}$

Explicit demonstration for rotations in Euclidean 2-space:

$$x^{1'} = \cos\theta\, x^1 + \sin\theta\, x^2 \qquad x^1 = \cos\theta\, x^{1'} - \sin\theta\, x^{2'}$$

$$x^{2'} = -\sin\theta\, x^1 + \cos\theta\, x^2 \qquad x^2 = \sin\theta\, x^{1'} + \cos\theta\, x^{2'}$$

$$\frac{\partial x^{1'}}{\partial x^1} = \cos\theta = \frac{\partial x^1}{\partial x^{1'}} \qquad \frac{\partial x^{2'}}{\partial x^1} = -\sin\theta = \frac{\partial x^1}{\partial x^{2'}}$$

$$\frac{\partial x^{1'}}{\partial x^2} = \sin\theta = \frac{\partial x^2}{\partial x^{1'}} \qquad \frac{\partial x^{2'}}{\partial x^2} = \cos\theta = \frac{\partial x^2}{\partial x^{2'}}$$

What about vectors in Minkowski space?

$$x^{1'} = \gamma x^1 - \gamma\beta x^4 \qquad\qquad x^1 = \gamma x^{1'} + \gamma\beta x^{4'}$$

$$x^{2'} = x^2 \qquad\qquad x^2 = x^{2'}$$

$$x^{3'} = x^3 \qquad\qquad x^3 = x^{3'}$$

$$x^{4'} = -\gamma\beta x^1 + \gamma x^4 \qquad\qquad x^4 = \gamma\beta x^{1'} + \gamma x^{4'}$$

$$\frac{\partial x^{1'}}{\partial x^4} = -\gamma\beta \quad \text{but} \quad \frac{\partial x^4}{\partial x^{1'}} = \gamma\beta$$

=> contravariant and covariant vectors are different!

Recap (for arbitrary space and transformation)

Contravariant vector: $A^{i'} = \sum_j \frac{\partial x^{i'}}{\partial x^j} A^j = \sum_j p^{i'}_j A^j$

Covariant vector: $A_{i'} = \sum_j \frac{\partial x^j}{\partial x^{i'}} A_j = \sum_j p^j_{i'} A_j$

For future convenience, define new notation for partial derivatives:

$$p^{i'}_i \equiv \frac{\partial x^{i'}}{\partial x^i} \quad ; \quad p^i_{i'} \equiv \frac{\partial x^i}{\partial x^{i'}} \quad ; \quad \frac{\partial^2 x^{i'}}{\partial x^i \partial x^j} = p^{i'}_{ij}$$

Note: $p^i_{i''} = \sum_{i'} p^i_{i'} p^{i'}_{i''} \quad : \quad \sum_{i'} p^i_{i'} p^{i'}_j = \delta^i_j$

δ^i_j = Kronecker delta = 1 if $i=j$, 0 if $i \neq j$

Tensors

Consider an N-dimensional space (with arbitrary geometry) and an object with components $A^{i\ldots k}_{l\ldots n}$ in the $\{x^i\}$ coord system and $A^{i'\ldots k'}_{l'\ldots n'}$ in the $\{x^{i'}\}$ coord system.

This object is a mixed tensor, contravariant in $i...k$ and covariant in $l...n$, under the coord transformation $\{x^i\} \rightarrow \{x^{i'}\}$ if

$$A^{i'\ldots k'}_{l'\ldots n'} = \sum_{i\ldots k,l\ldots n} A^{i\ldots k}_{l\ldots n}\, p^{i'}_{i} \ldots p^{k'}_{k}\, p^{l}_{l'} \ldots p^{n}_{n'}$$

Rank of tensor, M = number of indices

Total number of components = N^M

Vectors are first rank tensors and scalars are zero rank tensors.

If space is Euclidean N-space and transformation is rotation of Cartesian coords, then tensor is called a “Cartesian tensor”.

In Minkowski space and under Poincaré transformations, tensors are “Lorentz tensors”, or, “4-tensors”.

Zero tensor **0** has all its components zero in all coord systems.

Main theorem of tensor analysis:

If two tensors of the same type have all their components equal in one coord system, then their components are equal in all coord systems.

Einstein's summation convention: repeated upper and lower indices => summation

e.g.: $A_i B^i = \sum_{i=1}^{N} A_i B^i$

$A_i B^i$ could also be written $A_j B^j$; index is a "dummy index"

Another example: $A^{ij}_k B^k_j = \sum_{j=1}^{N} \sum_{k=1}^{N} A^{ij}_k B^k_j$

j and *k* are dummy indices; *i* is a "free index"

Summation convention also employed with $\frac{\partial u^i}{\partial x^i}$, $\frac{\partial q}{\partial x^i} \frac{dx^i}{d\tau}$, etc.

Example of a second rank tensor: Kronecker delta

$$\delta^i_j \, p^{i'}_i \, p^j_{j'} = p^{i'}_j \, p^j_{j'} = \delta^{i'}_{j'}$$

SP 5.1-2

Tensor Algebra (operations for making new tensors from old tensors)

1. Sum of two tensors: add components: $C^{i\ldots}_{k\ldots} = A^{i\ldots}_{k\ldots} + B^{i\ldots}_{k\ldots}$

   Proof that sum is a tensor: (for one case)

   $$C^{i'}_{k'} = A^{i'}_{k'} + B^{i'}_{k'} = A^{i}_{k}\, p^{i'}_{i}\, p^{k}_{k'} + B^{i}_{k}\, p^{i'}_{i}\, p^{k}_{k'}$$

   $$= \left(A^{i}_{k} + B^{i}_{k}\right) p^{i'}_{i}\, p^{k}_{k'} = C^{i}_{k}\, p^{i'}_{i}\, p^{k}_{k'}$$

2. Outer product: multiply components: e.g., $C^{ij}_{klm} = A^{i}_{k}\, B^{j}_{lm}$

3. Contraction: replace one superscript and one subscript by a dummy index pair

   e.g., $B^{j}_{km} = A^{hj}_{khm}$

   Result is a scalar if no free indices remain.

   e.g, A^{i}_{i} , A^{ij}_{ij} , $\delta^{i}_{i} = N$

4. Inner product: contraction in conjunction with outer product

e.g.: $C_{ikl} = A_{ij}\, B^{j}_{kl}$

Again, result is a scalar if no free indices remain, e.g, $A_{ij}\, B^{ij}$

5. Index permutation: e.g., $B_{ijk} = A_{ikj}$

SP 5.3-5

Differentiation of Tensors

Notation: $A^{i\ldots k}_{l\ldots n,r} \equiv \dfrac{\partial}{\partial x^r}\left(A^{i\ldots k}_{l\ldots n}\right)$; $A^{i\ldots k}_{l\ldots n,rs} \equiv \dfrac{\partial^2}{\partial x^r \partial x^s}\left(A^{i\ldots k}_{l\ldots n}\right)$, etc.

$$A^{i'\ldots k'}_{l'\ldots n',r'} = \frac{\partial}{\partial x^{r'}}\left(A^{i\ldots k}_{l\ldots n}\, p^{i'}_{i} \ldots p^{k'}_{k}\, p^{l}_{l'} \ldots p^{n}_{n'}\right)$$

$$= \frac{\partial}{\partial x^{r}}\left(A^{i\ldots k}_{l\ldots n}\, p^{i'}_{i} \ldots p^{k'}_{k}\, p^{l}_{l'} \ldots p^{n}_{n'}\right)\, p^{r}_{r'}$$

$$= A^{i\ldots k}_{l\ldots n,r}\, p^{i'}_{i} \ldots p^{k'}_{k}\, p^{l}_{l'} \ldots p^{n}_{n'} p^{r}_{r'}$$

IF transformation is linear
(so that p's are all constant)

=> derivative of a tensor wrt a coordinate is a tensor only for linear transformations (like rotations and LTs)

Similarly, differentiation wrt a scalar (e.g., τ) yields a tensor for linear transformations.

Now specialize to Riemannian spaces

characterized by a metric $ds^2 = g_{ij}\, dx^i\, dx^j$ with $\det(g_{ij}) \neq 0$

Assume g_{ij} is symmetric: $g_{ij} = g_{ji}$ (no loss of generality, since they only appear in pairs)

If $ds^2 > 0$ when $dx^i \not\equiv 0$, then space is "strictly Riemannian" (e.g., Euclidean *N*-space)

Otherwise, space is "pseudo-Riemannian" (e.g., Minkowski space)

g_{ij} is called the "metric tensor".

Note that the metric tensor may be a function of position in the space.

Proof that g_{ij} is a tensor:

$$g_{ij}dx^i dx^j = g_{ij}dx^{k'} p^i_{k'} dx^{l'} p^j_{l'}$$ (since dx^i is a vector)

$$ds^2 = g_{ij}dx^i dx^j = g_{k'l'}dx^{k'} dx^{l'}$$ (2 sets of dummy indices)

=> $$(g_{k'l'} - g_{ij}p^i_{k'}p^j_{l'})dx^{k'} dx^{l'} = 0$$

It's tempting to divide by $dx^{k'} dx^{l'}$ and conclude $g_{k'l'} = g_{ij}p^i_{k'}p^j_{l'}$

But there's a double sum over k' and l', so this isn't possible.

Instead, suppose $dx^{i'}$ = 1 if i' = 1
= 0 otherwise

=> $g_{1'1'} - g_{ij}\, p^i_{1'}\, p^j_{1'} = 0$ Similarly for $g_{2'2'}$, etc.

$$(g_{k'l'} - g_{ij}p^i_{k'}p^j_{l'})dx^{k'}dx^{l'} = 0$$

Now suppose $dx^{i'}$ = 1 if i' = 1 or 2
= 0 otherwise

Only contributing terms are: k'=1, l'=1 k'=1, l'=2
k'=2, l'=1 k'=2, l'=2

$$(g_{k'l'} - g_{ij}p^i_{k'}p^j_{l'})dx^{k'}dx^{l'} = \underbrace{g_{1'1'} - g_{ij}\,p^i_{1'}\,p^j_{1'}}_{0} + \underbrace{g_{2'2'} - g_{ij}\,p^i_{2'}\,p^j_{2'}}_{0} +$$

$$g_{1'2'} - g_{ij}\,p^i_{1'}\,p^j_{2'} + g_{2'1'} - g_{ij}\,p^i_{2'}\,p^j_{1'}$$

$g_{1'2'} = g_{2'1'}$ since g_{ij} is symmetric.

$g_{ij}\,p^i_{2'}\,p^j_{1'} = g_{ij}\,p^i_{1'}\,p^j_{2'}$ since i and j are dummy indices.

=> $2\left(g_{1'2'} - g_{ij}\,p^i_{1'}\,p^j_{2'}\right) = 0$ Similarly for all $g_{i'j'}$ $(i' \neq j')$

General definition of the scalar product: $\mathbf{A} \cdot \mathbf{B} = g_{ij} A^i B^j$

Define g^{ij} as the inverse matrix of g_{ij} : $g^{ij} g_{jk} = \delta^i_k$

g^{ij} is also a tensor, since applying tensor transformation yields

$g^{i'j'} g_{j'k'} = \delta^{i'}_{k'}$, which defines $g^{i'j'}$ as the inverse of $g_{i'j'}$

Raising and lowering of indices: another tensor algebraic operation, defined for Riemannian spaces = inner product of a tensor with the metric tensor

e.g.: $A_i = g_{ij} A^j$; $A^i = g^{ij} A_j$; $A^i{}_{jk} = g^{ir} g_{ks} A_{rj}{}^s$

Note: covariant and contravariant indices must be staggered when raising and lowering is anticipated.

4-tensors

In all coord systems in Minkowski space:

$$ds^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

=> $g_{\mu\nu} = \mathrm{diag}(-1,-1,-1,1) = g^{\mu\nu}$

e.g: $A_i = g_{i\mu}\, A^\mu = -A^i \quad (i = 1,2,3)$

$$A_4 = g_{4\mu} A^\mu = A^4$$

$$U^\mu = \gamma(u)\,(\mathbf{u}, c) \;\Rightarrow\; U_\mu = \gamma(u)\,(-\mathbf{u}, c)$$

Under standard Lorentz transformations:

$$p_1^{1'} = p_4^{4'} = \gamma \ , \quad p_4^{1'} = p_1^{4'} = -\gamma\beta \ , \quad p_2^{2'} = p_3^{3'} = 1$$

$$p_{1'}^{1} = p_{4'}^{4} = \gamma \ , \quad p_{4'}^{1} = p_{1'}^{4} = \gamma\beta \ , \quad p_{2'}^{2} = p_{3'}^{3} = 1$$

All the other *p*'s are zero.

e.g.: $$A^{1'2'} = A^{\mu\nu} p_\mu^{1'} p_\nu^{2'} = A^{\mu 2} p_\mu^{1'} = \gamma \left(A^{12} - \beta A^{42} \right)$$

SP 5.6-10