Electrodynamics

Recall Lorentz force law: $\mathbf{f} = q\left(\mathbf{e} + \frac{\mathbf{u}\times\mathbf{b}}{c}\right)$

e = electric field
b = magnetic field

f depends linearly on q and **u**, but is not parallel to **u**.

Simplest relativistic expression with these properties: $F_\mu = \frac{q}{c} E_{\mu\nu} U^\nu$

$E_{\mu\nu}$ is called the "electromagnetic field tensor"; q is assumed invariant.

We expect F_μ to be rest mass-preserving, which requires:

$\mathbf{F}\cdot\mathbf{U} = F_\mu U^\mu = 0$ => $E_{\mu\nu} U^\mu U^\nu = 0$ for all **U**

$$\Rightarrow E_{\mu\nu} = -E_{\nu\mu}$$

i.e., $E_{\mu\nu}$ is antisymmetric

SP 6.1

The Lorentz force law works out exactly if we take:

$$E_{\mu\nu} = \begin{pmatrix} 0 & -b_3 & b_2 & -e_1 \\ b_3 & 0 & -b_1 & -e_2 \\ -b_2 & b_1 & 0 & -e_3 \\ e_1 & e_2 & e_3 & 0 \end{pmatrix} \qquad E^{\mu\nu} = \begin{pmatrix} 0 & -b_3 & b_2 & e_1 \\ b_3 & 0 & -b_1 & e_2 \\ -b_2 & b_1 & 0 & e_3 \\ -e_1 & -e_2 & -e_3 & 0 \end{pmatrix}$$

From SP 5.10: **e** transforms as a 3-vector under spatial rotations.
b does, too (as we'll see later).

$$F_\mu = \frac{q}{c} E_{\mu\nu} U^\nu = \frac{q}{c}\gamma(u)\, E_{\mu\nu}\, (\mathbf{u}, c)^\nu = \frac{q}{c}\gamma(u)\, [-c\,\mathbf{e} - \mathbf{u}\times\mathbf{b}, \mathbf{e}\cdot\mathbf{u}]_\mu$$

Recall: $\mathbf{F} = \gamma(u)\left(\mathbf{f}, \frac{1}{c}\frac{dE}{dt}\right) \quad \Rightarrow F_\mu = \gamma(u)\left(-\mathbf{f}, \frac{1}{c}\frac{dE}{dt}\right)_\mu$

$$\Rightarrow \mathbf{f} = q\left(\mathbf{e} + \frac{\mathbf{u}\times\mathbf{b}}{c}\right)$$, i.e., the Lorentz force law!

Explicit demonstration that $E_{\mu\nu}\left(\mathbf{u},c\right)^{\nu} = \left[-c\,\mathbf{e} - \mathbf{u}\times\mathbf{b}, \mathbf{e}\cdot\mathbf{u}\right]_{\mu}$

This particular multiplication and sum is equivalent to a matrix multiplication:

$$\begin{pmatrix} 0 & -b_3 & b_2 & -e_1 \\ b_3 & 0 & -b_1 & -e_2 \\ -b_2 & b_1 & 0 & -e_3 \\ e_1 & e_2 & e_3 & 0 \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \\ u_3 \\ c \end{pmatrix} = \begin{pmatrix} -u_2b_3 + u_3b_2 - ce_1 \\ u_1b_3 - u_3b_1 - ce_2 \\ -u_1b_2 + u_2b_1 - ce_3 \\ u_1e_1 + u_2e_2 + u_3e_3 \end{pmatrix}$$

$$= \left(-c\mathbf{e} - \mathbf{u}\times\mathbf{b}, \mathbf{e}\cdot\mathbf{u}\right)$$

Maxwell's eqn's:

$$\nabla \cdot \mathbf{e} = 4\pi\rho \qquad \nabla \times \mathbf{e} = -\frac{1}{c}\frac{\partial \mathbf{b}}{\partial t}$$

$$\nabla \cdot \mathbf{b} = 0 \qquad \nabla \times \mathbf{b} = \frac{4\pi}{c}\mathbf{j} + \frac{1}{c}\frac{\partial \mathbf{e}}{\partial t}$$

Relativistic sources:

ρ_0 = proper charge density (i.e., in frame comoving with the charge)

$\rho = \rho_0\, \gamma(u)$ total $\rho = \sum_i \rho_{0,i}\, \gamma(u_i)$

Note: sum is over each moving charge dist.; *i* is not an index!

3-current density: $\mathbf{j}_i = \rho_i \mathbf{u}_i \quad ; \quad \mathbf{j} = \sum_i \mathbf{j}_i$

4-current density:

$$J_i^\mu = \rho_{0,i}\, U_i^\mu = \rho_{0,i}\, \gamma(u_i)\, (\mathbf{u}_i, c) = (\mathbf{j}_i, c\rho_i)$$

$$J^\mu = \sum_i J_i^\mu = (\mathbf{j}, c\rho)$$

$$J^{\mu} = (\mathbf{j}, c\rho)$$

Recall the continuity eqn: $\nabla \cdot \mathbf{j} = -\frac{\partial \rho}{\partial t}$

In 4-tensor notation: $J^{\mu}{}_{,\mu} = 0$

$$\begin{aligned} J^{\mu}{}_{,\mu} &= J^{1}{}_{,1} + J^{2}{}_{,2} + J^{3}{}_{,3} + J^{4}{}_{,4} \\ &= \frac{\partial j_1}{\partial x_1} + \frac{\partial j_2}{\partial x_2} + \frac{\partial j_3}{\partial x_3} + \frac{\partial (c\rho)}{\partial (ct)} \\ &= \nabla \cdot \mathbf{j} + \frac{\partial \rho}{\partial t} \end{aligned}$$

4-tensor form of Maxwells' eqns:

$$E^{\mu\nu}{}_{,\mu} = \frac{4\pi}{c} J^{\nu}$$

$$E_{\mu\nu,\sigma} + E_{\nu\sigma,\mu} + E_{\sigma\mu,\nu} = 0$$

$$E^{\mu\nu}{}_{,\mu} = \frac{4\pi}{c} J^{\nu}$$ (4 equations)

ν = 1: $$E^{11}{}_{,1} + E^{21}{}_{,2} + E^{31}{}_{,3} + E^{41}{}_{,4} = \frac{4\pi}{c} J^{1}$$

$$0 + \frac{\partial b_3}{\partial x_2} - \frac{\partial b_2}{\partial x_3} - \frac{1}{c}\frac{\partial e_1}{\partial t} = \frac{4\pi}{c} j^{1}$$

1st component of: $$\nabla \times \mathbf{b} - \frac{1}{c}\frac{\partial \mathbf{e}}{\partial t} = \frac{4\pi}{c}\mathbf{j}$$ (similarly for ν = 2, 3)

ν = 4: $$E^{14}{}_{,1} + E^{24}{}_{,2} + E^{3,4}{}_{,3} + E^{44}{}_{,4} = \frac{4\pi}{c} J^{4}$$

$$\frac{\partial e_1}{\partial x_1} + \frac{\partial e_2}{\partial x_2} + \frac{\partial e_3}{\partial x_3} + 0 = \frac{4\pi}{c} c\,\rho$$

i.e.: $$\nabla \cdot \mathbf{e} = 4\pi\rho$$

$$E_{\mu\nu,\sigma} + E_{\nu\sigma,\mu} + E_{\sigma\mu,\nu} = 0$$ (64 equations)

μ=1, ν=2, σ=3:

$$\begin{aligned} E_{12,3} + E_{23,1} + E_{31,2} &= 0 \\ -\frac{\partial b_3}{\partial x_3} - \frac{\partial b_1}{\partial x_1} - \frac{\partial b_2}{\partial x_2} &= 0 \end{aligned} \qquad \nabla \cdot \mathbf{b} = 0$$

μ, ν, σ = 2, 3, 4:

$$\begin{aligned} E_{23,4} + E_{34,2} + E_{42,3} &= 0 \\ -\frac{1}{c}\frac{\partial b_1}{\partial t} - \frac{\partial e_3}{\partial x_2} + \frac{\partial e_2}{\partial x_3} &= 0 \end{aligned}$$

1st component of: $$-\frac{1}{c}\frac{\partial \mathbf{b}}{\partial t} - \nabla \times \mathbf{e} = 0$$

μ, ν, σ = 3, 4, 1 ; 4, 1, 2 yield 2nd and 3rd components.

Other combinations of μ, ν, σ yield repeats of old eqns or 0 = 0.

Note: $E^{\mu\nu}{}_{,\mu} = \frac{4\pi}{c} J^{\nu}$ => $J^{\mu}{}_{,\mu} = 0$ (continuity eqn)

since $E^{\mu\nu}{}_{,\mu\nu}$ is symmetric in its subscripts (equality of mixed partial derivs) but antisymmetric in its superscripts.

The field tensor can be expressed in terms of a 4-potential Φ_{μ} :

$$E_{\mu\nu} = \Phi_{\nu,\mu} - \Phi_{\mu,\nu}$$

This definition immediately yields the 2nd tensor field eqn:

$$E_{\mu\nu,\sigma} + E_{\nu\sigma,\mu} + E_{\sigma\mu,\nu}$$

$$= \Phi_{\nu,\mu\sigma} - \Phi_{\mu,\nu\sigma} + \Phi_{\sigma,\nu\mu} - \Phi_{\nu,\sigma\mu} + \Phi_{\mu,\sigma\nu} - \Phi_{\sigma,\mu\nu} = 0$$

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SP 6.2

Notes: The field does not *uniquely* determine the potential.

Φ_μ is a 4-vector (if we pick it in one IF, then its transforms in all other IFs will yield the field in those IFs)

1st tensor field eqn: $E^{\mu\lambda}{}_{,\mu} = \frac{4\pi}{c} J^\lambda$

$$g_{\nu\lambda} E^{\mu\lambda}{}_{,\mu} = \frac{4\pi}{c} g_{\nu\lambda} J^\lambda$$

$$E^{\mu}{}_{\nu,\mu} = \frac{4\pi}{c} J_\nu$$

$$E^{\mu}{}_{\nu,\mu} = g^{\mu\sigma} E_{\sigma\nu,\mu} = g^{\mu\sigma} \left(\Phi_{\nu,\sigma\mu} - \Phi_{\sigma,\nu\mu}\right)$$

$$\Rightarrow g^{\mu\sigma} \left(\Phi_{\nu,\sigma\mu} - \Phi_{\sigma,\nu\mu}\right) = \frac{4\pi}{c} J_\nu$$

It is possible to choose Φ_μ such that $\Phi^{\mu}{}_{,\mu} = 0$ (see Rindler 7.4)

(called the "Lorenz gauge condition")

With this choice, the 2nd term in the field eqn vanishes:

$$g^{\mu\sigma}\Phi_{\sigma,\nu\mu} = \Phi^{\mu}{}_{,\nu\mu} = \Phi^{\mu}{}_{,\mu\nu} = \left(\Phi^{\mu}{}_{,\mu}\right)_{,\nu} = 0_{,\nu} = 0$$

Leaves us with: $g^{\mu\sigma}\,\Phi_{\nu,\sigma\mu} = \frac{4\pi}{c}\,J_\nu$, or, $g^{\mu\sigma}\,\Phi_{\nu,\mu\sigma} = \frac{4\pi}{c}\,J_\nu$

$_{,\mu\nu}\,g^{\mu\nu}$ is a differential operator:

$$_{,\mu\nu}\,g^{\mu\nu} = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2} - \frac{\partial^2}{\partial y^2} - \frac{\partial^2}{\partial z^2}$$

$$= \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 = \Box \quad\longrightarrow\quad \Box\,\Phi_\mu = \frac{4\pi}{c}\,J_\mu$$

(the d'Alembertian)

In vacuum, $J_\mu = 0$ => $\Box\, \Phi_\mu = 0$ (the wave equation)

=> wave eqn for the field, too:

$$\begin{aligned} \Box E_{\mu\nu} &= g^{\sigma\lambda}\left(\Phi_{\nu,\mu\sigma\lambda} - \Phi_{\mu,\nu\sigma\lambda}\right) \\ &= \left(g^{\sigma\lambda}\Phi_{\nu,\sigma\lambda}\right)_{,\mu} - \left(g^{\sigma\lambda}\Phi_{\mu,\sigma\lambda}\right)_{,\nu} \\ &= \left(\Box\Phi_\nu\right)_{,\mu} - \left(\Box\Phi_\mu\right)_{,\nu} \\ &= 0_{,\mu} - 0_{,\nu} = 0 \end{aligned}$$

Transformation of **e** and **b** under a standard LT:

SP 6.3

$$e_{1'} = e_1 \quad , \quad e_{2'} = \gamma(e_2 - \beta\, b_3) \quad , \quad e_{3'} = \gamma(e_3 + \beta\, b_2)$$

$$b_{1'} = b_1 \quad , \quad b_{2'} = \gamma(b_2 + \beta\, e_3) \quad , \quad b_{3'} = \gamma(b_3 - \beta\, e_2)$$

Note symmetry: transformation is unchanged if **b** → **e** and **e** → - **b** :

$$-b_{1'} = -b_1 \quad , \quad -b_{2'} = \gamma(-b_2 - \beta\, e_3) \, , \quad -b_{3'} = \gamma(-b_3 + \beta\, e_2)$$

$$e_{1'} = e_1 \quad , \quad e_{2'} = \gamma(e_2 - \beta\, b_3) \quad , \quad e_{3'} = \gamma(e_3 + \beta\, b_2)$$

If we write the field transformation as (**e**′, **b**′) = T (**e**, **b**),

then: (-**b**′, **e**′) = T (-**b**, **e**).

Form the dual of $E_{\mu\nu}$ by interchanging components; **e** →-**b**, **b**→**e** :

$$B_{\mu\nu} = \begin{pmatrix} 0 & -e_3 & e_2 & b_1 \\ e_3 & 0 & -e_1 & b_2 \\ -e_2 & e_1 & 0 & b_3 \\ -b_1 & -b_2 & -b_3 & 0 \end{pmatrix} \qquad B^{\mu\nu} = \begin{pmatrix} 0 & -e_3 & e_2 & -b_1 \\ e_3 & 0 & -e_1 & -b_2 \\ -e_2 & e_1 & 0 & -b_3 \\ b_1 & b_2 & b_3 & 0 \end{pmatrix}$$

$B_{\mu\nu}$ is a tensor, since its components transform in the same way as those of $E_{\mu\nu}$. (=> **b** transforms as a 3-vector under spatial rotations)

Maxwell's equations can be written in the form:

$$E^{\mu\nu}{}_{,\mu} = \frac{4\pi}{c} J^{\nu} \qquad B^{\mu\nu}{}_{,\mu} = 0$$

There are two invariants associated with $E_{\mu\nu}$:

$$X = \frac{1}{2} E_{\mu\nu} E^{\mu\nu} = \mathbf{b}^2 - \mathbf{e}^2 = -\frac{1}{2} B_{\mu\nu} B^{\mu\nu}$$

$$Y = \frac{1}{4} E_{\mu\nu} B^{\mu\nu} = \mathbf{e} \cdot \mathbf{b}$$

=> if electric and magnetic fields have equal magnitude in one IF, then they have equal magnitudes in all IFs.

if they're orthogonal in one IF, then they're orthogonal in all IFs.

SP 6.4

Example #1: the field of a uniformly moving point charge

Suppose charge q is moving at speed βc along the $+x$-axis and is at the origin at $t = 0$.

In q's rest frame: $\mathbf{e}' = \frac{q}{r'^3}(x', y', z')$ $\mathbf{b}' = 0$

$$r'^2 = x'^2 + y'^2 + z'^2$$

v-reversed transformation law for $\mathbf{e}$:

$$e_1 = e_{1'} \quad , \quad e_2 = \gamma(e_{2'} + \beta\, b_{3'}) \quad , \quad e_3 = \gamma(e_{3'} - \beta\, b_{2'})$$

=> $e_1 = e_{1'} \quad , \quad e_2 = \gamma e_{2'} \quad , \quad e_3 = \gamma e_{3'}$

LT with $t = 0$: $x = x'/\gamma \quad , \quad y = y' \quad , \quad z = z'$

=> $e_1 = \frac{\gamma\, q\, x}{r'^3} \quad , \quad e_2 = \frac{\gamma\, q\, y}{r'^3} \quad , \quad e_3 = \frac{\gamma\, q\, z}{r'^3}$

$$\mathbf{e} = \frac{\gamma\, q\, \mathbf{r}}{r'^{\,3}} \qquad\qquad r'^{\,2} = \gamma^2 x^2 + y^2 + z^2 \;=\; \gamma^2 r^2 - (\gamma^2 - 1)(y^2 + z^2)$$

$$= \gamma^2 r^2 - \gamma^2 \beta^2 (y^2 + z^2)$$

$$= \gamma^2 r^2 (1 - \beta^2 \sin^2 \theta)$$

θ = angle btwn *x*-axis and **r**

=> $$\mathbf{e} = \frac{q\, \mathbf{r}}{\gamma^2 r^3 (1 - \beta^2 \sin^2 \theta)^{3/2}}$$

transformation eqns for **b** :

$$b_{1'} = b_1 \quad , \quad b_{2'} = \gamma(b_2 + \beta\, e_3) \quad , \quad b_{3'} = \gamma(b_3 - \beta\, e_2)$$

with **b**′ = 0 : $b_1 = 0 \quad , \quad b_2 = -\beta e_3 \quad , \quad b_3 = \beta e_2$

=> $$\mathbf{b} = \frac{1}{c}(0, -v_1 e_3, v_1 e_2) = \frac{\mathbf{v} \times \mathbf{e}}{c}$$

Example #2: the field of an infinite straight current

Consider a line with rest-frame charge density λ_0, moving in positive direction in frame S along x-axis with speed v.

In line's rest frame, Gauss's Law yields: $\mathbf{e}' = \frac{2\lambda_0}{r'}\hat{r}'$

In S: length contraction yields $\lambda = \gamma\,\lambda_0$; current $i = \lambda\, v$

Consider the point $(x = 0,\ y = r,\ z = 0)$. In S′: $(x' = 0,\ y' = r,\ z' = 0)$

$$\mathbf{e}' = (0, 2\lambda_0/r, 0) \qquad \mathbf{b}' = 0$$

$$e_{1'} = e_1 \quad , \quad e_{2'} = \gamma(e_2 - \beta\, b_3) \quad , \quad e_{3'} = \gamma(e_3 + \beta\, b_2)$$
$$b_{1'} = b_1 \quad , \quad b_{2'} = \gamma(b_2 + \beta\, e_3) \quad , \quad b_{3'} = \gamma(b_3 - \beta\, e_2)$$

with a v-reversal

=> $\mathbf{e} = (0, 2\gamma\lambda_0/r, 0) \qquad \mathbf{b} = (0, 0, 2\gamma\beta\lambda_0/r)$

or: $\mathbf{e} = \frac{2\lambda}{r}\hat{r} \qquad \mathbf{b} = \frac{2\lambda v}{cr} = \frac{2i}{cr}(\hat{i} \times \hat{r})$

In a real wire: stationary positive charges (ions) have same λ as the electrons; electric fields cancel and only magnetic field remains.

From the point of view of a moving charged particle:

In its rest frame, only electric fields will produce a force.

It sees two electric fields from the wire, due to (a) the nuclei and (b) the electrons.

Because of length contraction, one of the charge densities exceeds the other and there is a net electric field; what we in frame S call a magnetic field.

=> magnetic deflection of a charged particle near a current carrying wire is a manifestation of length contraction, even though the electron speed in the wire is only ~ mm/s!

SP 6.5