# Introduction to General Relativity

Recall Newtonian gravitation: $\mathbf{F}_g = -\nabla\phi \quad ; \quad \nabla^2\phi = 4\pi G\rho$

Clearly not Lorentz-invariant, since Laplacian appears rather than d'Alembertian. No attempt to find Lorentz-invariant equations that reduce to Newtonian version in appropriate limit has succeeded.

A thought experiment (in 2 parts) suggests that SR and gravitation are inherently incompatible.

## Part 1:

a) On Earth, drop a ball of rest mass $m_0$ from height $h$.

b) When ball reaches ground:

$$mc^2 = m_0c^2 + T \approx m_0c^2 + \frac{1}{2}m_0v^2 = m_0c^2 + m_0gh$$

c) Convert entire mass of ball to a photon and send it straight up:

$$h\nu_{\text{bot}} = E_{\text{bot}} \approx m_0 c^2 + m_0 g h$$

d) When photon reaches height *h*, convert it into a ball.

Conservation of energy => $h\nu_{\text{top}} = m_0 c^2$

e) $$\frac{\nu_{\text{top}}}{\nu_{\text{bot}}} = \frac{m_0 c^2}{m_0 c^2 + m_0 g h} = \left(1 + \frac{gh}{c^2}\right)^{-1} \approx 1 - \frac{gh}{c^2}$$

=> gravitational redshift

Part 2: Within SR, consider the IF attached to Earth
(suppose Earth doesn't rotate, orbit, etc.)

In this IF, stationary clocks at different spatial locations run at same rate.

$\nu^{-1}$ = time interval between wave crests; different for observers on ground and at height $h$

=> stationary clocks at different locations run at different rates

=> Our IF is not an IF after all!

=> We must modify SR and/or our notion of gravitation.

As with SR, a big coincidence in classical physics guided Einstein in constructing GR: equivalence of gravitational and inertial mass (known as the weak equivalence principle)

Inertial mass: $\mathbf{F} = m_i\, \mathbf{a}$

Gravitational mass: $\mathbf{F}_{\mathrm{g}} = G\, m_{g1} m_{g2} / r^2$

Further distinguish between active gravitational mass (which establishes the field) and passive gravitational mass (which responds to the field).

WEP: Gravitational force is proportional to the inertial mass (in contrast to, e.g., the electric force, which depends on $q$)

Forces that are proportional to the inertial mass also arise when an observer adopts a non-inertial reference frame. These are known as inertial forces (or pseudo-forces, fictitious forces, kinematic forces).

Consider a frame S' moving with constant acceleration $\mathbf{a}_0$

$\mathbf{a} = \mathbf{a}_0 + \mathbf{a}'$ ($\mathbf{a}$ = accel rel to an IF; $\mathbf{a}'$ = accel rel to S')

$\mathbf{F} = m\,\mathbf{a} = m\,(\mathbf{a}_0 + \mathbf{a}')$ => $\mathbf{F} - m\,\mathbf{a}_0 = m\,\mathbf{a}'$

For observer in S', Newton's 2nd Law holds and every body with mass $m$ is subjected to a force $-\,m\,\mathbf{a}_0$

(Similarly, centrifugal and Coriolis forces arise when observer adopts a uniformly rotating frame.)

Observer in an IF: Inertial forces do not exist; observer in S' does not correctly identify free particles.

Accelerometers carried along with the observed bodies will agree with the observer in the IF.

Einstein: Gravity is an inertial force, too!

If gravity is really an inertial force, then we should be able to identify a frame in which the gravitational force is absent.

Ask the astronauts on the space station: we will now identify freely falling frames as IFs.

A free particle is one that falls freely under gravity; no “real” forces act on it.

Is this reasonable? Yes!

a) An accelerometer on a freely falling body reads zero, whereas those attached to bodies at “rest” on Earth's surface do not (just get on a scale!)

b) A freely falling observer sees no gravitational frequency shift; it is exactly compensated by the Doppler shift! So, for an observer in free fall in a uniform grav field, spatially separated clocks do run at the same rate.

Suppose the entire universe were permeated by a uniform gravitational field (and that were the only field).

There would be no way to detect this field, since all particles would respond to it in the same way (i.e., no accelerometer could be built).

=> we could retain the spacetime geometry of SR (i.e., Minkowski space)

But: the universe is full of non-uniform gravitational fields

=> we cannot construct the infinite-extent IFs of SR

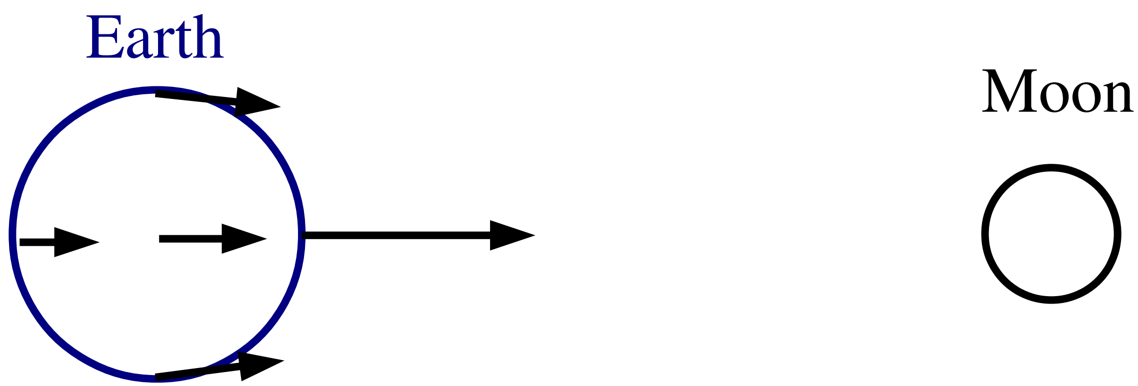
e.g., Earth: IFs fall in opposite directions on opposite sides of Earth.

=> the geometry of spacetime is more complicated than that of Minkowski space

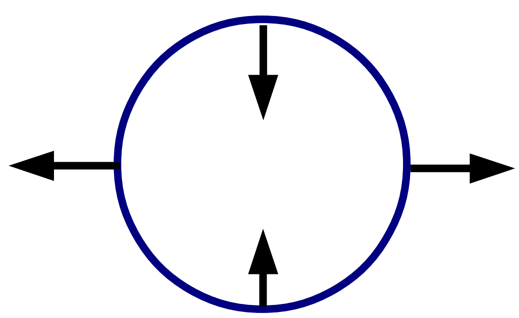
At each point in spacetime (i.e., at each event), we can identify local inertial reference frames (LIFs).

Need to make the idealization that the spatial and temporal extents of LIF go to zero; otherwise tidal forces are felt.

Tidal forces are differential gravitational forces; so named because they raise ocean tides:




In Earth's rest frame:



The geometry of Minkowski space, like that of Euclidean space, is flat.

Means that if the wordlines (curves in spacetime) of 2 free particles are initially parallel, then they remain so.

Because of non-uniform gravitational fields, 2 free particles moving parallel initially generally do not remain parallel (e.g., 2 particles approaching Sun from interstellar space).

=> Spacetime is not flat.

Example of a 2-D space that is not flat: the surface of a sphere

2 trajectories, each perpendicular to the equator, will approach one another; analogous to the tidal force.

Globally, the geometry of a sphere's surface is not Euclidean, but Euclidean geometry applies to high accuracy in any small region of the surface; geometry is approximately that of the tangent plane.

Analogously, the geometry of 4-D spacetime is not that of Minkowski space, but in any small region, Minkowski space is a good approximation (the LIFs are analogous to the tangent planes of the sphere).

Note:  we can't picture curvature of spaces with 3 or more dimensions, since we need an embedding space.  But, embedding space is not needed for a complete description of the geometry.

2-D beings on surface of sphere could determine that the geometry of their world is non-Euclidean, without ever experiencing the 3rd dimension.

# Summary

1. Spacetime is curved => there are no global IFs.

2. Locally, gravity can be “transformed away” by switching to a LIF. If a LIF is not adopted, a gravitational force appears; since it's an inertial force, a particle's passive grav mass equals its inertial mass.

3. Equivalence of active grav mass with inertial mass remains a mystery. Evidently spacetime curvature depends on the mass of the sources.

4. SR is “special” because it assumes the existence of a special class of observers, the inertial observers. In GR, extended IFs don't exist, so we need a theory valid for accelerating observers. This is how GR generalizes SR.

5. At each point in spacetime (i.e., at each event), the geometry is approximately that of Minkowski space => locally, free (i.e., freely falling) particles have straight worldlines.

6. Global worldlines of free particles are locally straight => they are "as straight as possible" in the curved spacetime. Analogy on the surface of a sphere: arc of a great circle. Such curves are known as "geodesics".

Our agenda for developing GR:

1. How can we describe the geometry of curved spacetime?
   (basic ideas of Riemannian geometry)

2. Given a spacetime geometry, how will free particles move?
   (mathematical description of geodesics)

3. How can we express the laws of physics in a curved spacetime?

Fundamental axiom is Einstein's equivalence principle (generalization of the weak EP to all of physics):

The laws of physics are the same in all LIFs. In particular, they are the same as in a universe with no spacetime curvature, i.e., they are the laws valid in SR. ("comma-goes-to-semicolon" rule)

4. How is the curvature of spacetime established?

(Einstein's field equations)