Geodesics in Curved Spacetime

Intrinsic vs. extrinsic curvature:

Extrinsic curvature depends on embedding space; can be eliminated by "unrolling", etc. (e.g., surface of a cylinder)

2-D beings on the surface of a sphere can tell that their world is not flat:

parallel transport around a circuit

relations for area, volume, etc. deviate from Euclidean relations (e.g.: great circle has $C = 4R < 2\pi R$)

Info on curvature is entirely contained within the metric tensor $g_{\mu\nu}$

(since it contains all info on distance relations)

We are only interested in intrinsic curvature of 4-D spacetime; we would have no way of probing any extrinsic curvature!

Riemannian spaces

Recall definition: characterized by metric tensor

$$d\mathbf{s}^2 = g_{\mu\nu}\, dx^\mu\, dx^\nu$$

Adopting "Cartesian" coords in Minkowski space, metric tensor is given by $\eta_{\mu\nu}$ = diag(-1, -1, -1, 1)

We must consider arbitrary (but smooth) coordinatizations of our space; Cartesian coords are not possible.

Theorem of linear algebra: Given a symmetric, invertible matrix (like $g_{\mu\nu}$), a transformation matrix can always be found to transform it into a diagonal matrix, with each element either +1 or -1.

Sum of these diagonal elements is called the "signature".

Signature of Euclidean N-space is N (metric tensor is δ_{ij})

Signature of Minkowski space is -2.

Equivalence Principle => appropriate choice of coord system will make $g_{\mu\nu} = \eta_{\mu\nu}$ at a single point in spacetime (yields the LIF)

=> signature of spacetime must everywhere be -2

Local flatness theorem: Given any point P in spacetime, a coord system can be found with its origin at P and with:

$$g_{\mu\nu}(x^\alpha) = \eta_{\mu\nu} + O[(x^\alpha)^2]$$ (called a "geodesic coord system")

i.e.: $g_{\mu\nu}(\mathrm{P}) = \eta_{\mu\nu}$ $\quad g_{\mu\nu,\alpha}(\mathrm{P}) = 0$ $\quad g_{\mu\nu,\alpha\beta} \neq 0$

To see this: $g_{\mu'\nu'} = g_{\mu\nu}\, p^{\mu}_{\mu'}\, p^{\nu}_{\nu'}$ ($g_{\mu\nu}$ is the metric in given coords and $g_{\mu'\nu'}$ is the metric we want, i.e., $\eta_{\mu\nu}$)

Taylor expand $g_{\mu\nu}$ and $p^{\mu}_{\mu'}$ about point $\mathbf{x}_0{}'$

Yields $g_{\mu'\nu'}$ = constant term + term linear in $d\mathbf{x}'$ + term quadratic in $d\mathbf{x}'$ + ...

Constant term involves $g_{\mu'\nu'}(\mathbf{x}_0')$ (=> 10 constraints) and

$p^{\mu}_{\mu'}(\mathbf{x}_0')$ (=> 16 free parameters)

=> 6 more parameters than we need (degrees of freedom for velocity and orientation of LIF)

Linear term involves $g_{\mu'\nu',\beta'}$ (=> 40 constraints) and

$p^{\mu}_{\mu'\beta'}$ (=> 40 free parameters)

=> linear term can be made to vanish (with no additional freedom)

Quadratic term involves $g_{\mu'\nu',\beta'\gamma'}$ (=> 100 constraints) and

$p^{\mu}_{\mu'\beta'\gamma'}$ (=> 80 free parameters)

=> we can't get the 2nd derivatives to all vanish; 20 numbers will arise at each point, characterizing the curvature

SP 8.1-2

Tensor analysis: We will now consider tensors in curved spacetime under general coord transformations (i.e., between arbitrary systems).

Recall: the only tensor operation we discussed relative to Minkowski space and Poincaré transformations that does not generalize to any Riemannian space and transformation is differentiation.

Since general coord transformations are not linear, partial differentiation of tensors will not yield new tensors; i.e., it's not a tensor operation.

Geodesics:

a) paths of stationary "length"

b) "straightest" possible paths

path = connected series of points

curve = parameterized path; coords of points on path are functions of the parameter u

Geodesic is a path of stationary (often extremal) length connecting 2 points A and B. If the path between A and B is varied somewhat, the geodesic is characterized by:

$$\delta \int ds = 0$$

=> $$\delta \int \left| g_{ij}\, dx^i\, dx^j \right|^{1/2} = \delta \int_{u_1}^{u_2} \left| g_{ij} \frac{dx^i}{du} \frac{dx^j}{du} \right|^{1/2} du = 0$$

(In the last step, we parameterized each path, with the same values u_1 and u_2 at A and B.)

Define $\dot{x}^i \equiv \dfrac{dx^i}{du}$ and $L(x^i, \dot{x}^i) \equiv \left| g_{ij}\, \dot{x}^i\, \dot{x}^j \right|^{1/2}$

Notes: 1. $\dot{x}^i$ is a vector (since dx^i transforms like dx^i and u is a scalar).

2. L is a function of $2N$ variables; x^i appears as well as $\dot{x}^i$ because g_{ij} depends on x^i.

We seek the $x^i(u)$ for which: $\delta \int_{u_1}^{u_2} L(x^i, \dot{x}^i)\, du = 0$

Solution is found by integrating the Euler-Lagrange eqns:

$$\frac{d}{du}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0$$

Derivation of Euler-Lagrange eqns:

Suppose geodesic is $x^i(u)$ and consider neighboring paths of form:

$$y^i(u) = x^i(u) + \epsilon\, w^i(u)$$

w^i are arbitrary functions with $w^i(u_1) = w^i(u_2) = 0$
and ϵ is a small real number.

Integral: $I(\epsilon) = \int_{u_1}^{u_2} L(y^i, \dot{y}^i)\, du = \int_{u_1}^{u_2} L(x^i + \epsilon\, w^i, \dot{x}^i + \epsilon\, \dot{w}^i)\, du$

$$I(\epsilon) = \int_{u_1}^{u_2} L(y^i, \dot{y}^i)\, du = \int_{u_1}^{u_2} L(x^i + \epsilon\, w^i, \dot{x}^i + \epsilon\, \dot{w}^i)\, du$$

Expand about $\epsilon = 0$:

$$I(\epsilon) = \int_{u_1}^{u_2} L(x^i, \dot{x}^i)\, du + \epsilon \int_{u_1}^{u_2} \left[\frac{\partial L}{\partial x^i} w^i + \frac{\partial L}{\partial \dot{x}^i} \dot{w}^i\right] du \quad + O(\epsilon^2)$$

$$\delta I(\epsilon) = 0 \Rightarrow \frac{dI}{d\epsilon} = 0 \Rightarrow \int_{u_1}^{u_2} \left[\frac{\partial L}{\partial x^i} w^i + \frac{\partial L}{\partial \dot{x}^i} \dot{w}^i\right] du = 0$$

Note: $$\frac{d}{du}\left[\frac{\partial L}{\partial \dot{x}^i}\, w^i\right] = \frac{\partial L}{\partial \dot{x}^i}\, \dot{w}^i + w^i\, \frac{d}{du}\left(\frac{\partial L}{\partial \dot{x}^i}\right)$$

=> $$\int_{u_1}^{u_2} \left[\frac{\partial L}{\partial x^i}\, w^i - w^i\, \frac{d}{du}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\right] du + \int_{u_1}^{u_2} d\left[\frac{\partial L}{\partial \dot{x}^i}\, w^i\right] = 0$$

this term = 0 since $w^i = 0$
at endpoints

=> $$\int_{u_1}^{u_2} w^i \left[\frac{\partial L}{\partial x^i} - \frac{d}{du}\left(\frac{\partial L}{\partial \dot{x}^i}\right)\right] du = 0$$

Must be true for arbitrary w^i => $$\frac{d}{du}\left(\frac{\partial L}{\partial \dot{x}^i}\right) - \frac{\partial L}{\partial x^i} = 0$$

It's always possible to parameterize the path with an affine parameter, i.e., one for which L is constant.

e.g.: massive particles move with $d\mathbf{s}^2 = c^2 dt^2 - d\mathbf{r}^2 > 0$

s is an affine parameter, with $L = 1$; geodesic is "timelike"

For light, $d\mathbf{s}^2 = 0$ => s cannot be chosen as an affine parameter.

"null", or "lightlike", geodesic

With $u = s$, $L = 1$ and we can replace L with $\mathcal{L} = \pm L^2 = g_{ij}\,\dot{x}^i\,\dot{x}^j$

=> $$\frac{d}{ds}\left(\frac{\partial \mathcal{L}}{\partial \dot{x}^i}\right) - \frac{\partial \mathcal{L}}{\partial x^i} = 0$$ with $$\mathcal{L} = \pm L^2 = g_{ij}\,\dot{x}^i\,\dot{x}^j$$

$$\frac{\partial \mathcal{L}}{\partial \dot{x}^i} = \frac{\partial(g_{jk}\,\dot{x}^j\,\dot{x}^k)}{\partial \dot{x}^i}$$

Now turn off the summation convention.

$$= \sum_{j \neq i} \frac{\partial\left(2 g_{ij} \dot{x}^i \dot{x}^j\right)}{\partial \dot{x}^i} + \frac{\partial\left[g_{ii}\left(\dot{x}^i\right)^2\right]}{\partial \dot{x}^i}$$

(Any terms that do not contain $\dot{x}^i$ do not survive the differentiation. So only consider terms where either j or k or both = i.)

$$= 2 \sum_{j \neq i} g_{ij} \dot{x}^j + 2 g_{ii} \dot{x}^i$$

$$= 2\,g_{ij}\,\dot{x}^j$$

(summation convention back on)

$$\frac{\partial \mathcal{L}}{\partial x^i} = \frac{\partial(g_{jk}\,\dot{x}^j\,\dot{x}^k)}{\partial x^i} = g_{jk,i}\,\dot{x}^j\,\dot{x}^k$$

=> $$\frac{d}{ds}(2\,g_{ij}\,\dot{x}^j) - g_{jk,i}\,\dot{x}^j\,\dot{x}^k = 0$$

From previous slide: $$\frac{d}{ds}(2\,g_{ij}\,\dot{x}^j) - g_{jk,i}\,\dot{x}^j\dot{x}^k = 0$$

$$\frac{d}{ds}(2\,g_{ij}\,\dot{x}^j) = 2\,g_{ij}\,\ddot{x}^j + 2\,\frac{dg_{ij}}{ds}\,\dot{x}^j \quad = 2\,g_{ij}\,\ddot{x}^j + 2\,g_{ij,k}\,\dot{x}^j\,\dot{x}^k$$

$$\Rightarrow 2\,g_{ij}\,\ddot{x}^j + 2\,g_{ij,k}\,\dot{x}^j\,\dot{x}^k - g_{jk,i}\,\dot{x}^j\,\dot{x}^k = 0$$

$$(g_{ij,k} + g_{ij,k} - g_{jk,i})\,\dot{x}^j\,\dot{x}^k + 2\,g_{ij}\,\ddot{x}^j = 0$$

j and *k* are dummy indices; reverse them in the 2nd term:

$$(g_{ij,k} + g_{ik,j} - g_{jk,i})\,\dot{x}^j\,\dot{x}^k + 2\,g_{ij}\,\ddot{x}^j = 0$$

Define Christoffel symbol of the 1st kind:

$$\Gamma_{ijk} \equiv \frac{1}{2}\,(g_{ij,k} + g_{ik,j} - g_{jk,i})$$

$$\Rightarrow 2\,\Gamma_{ijk}\,\dot{x}^j\,\dot{x}^k + 2\,g_{ij}\,\ddot{x}^j = 0$$

Rename free index $i \to h$: $\Gamma_{hjk}\dot{x}^j\dot{x}^k + g_{hj}\ddot{x}^j = 0$

Multiply by g^{hi} and define $\Gamma^i_{jk} \equiv g^{hi}\,\Gamma_{hjk}$ (Christoffel symbol of the 2nd kind, or, "connection coefficient")

$$\Gamma^i_{jk}\,\dot{x}^j\,\dot{x}^k + g^i{}_j\,\ddot{x}^j = 0$$

Recall $g^i{}_j = \delta^i{}_j$

=> $\boxed{\ddot{x}^i + \Gamma^i_{jk}\,\dot{x}^j\,\dot{x}^k = 0}$ (the geodesic equation)

Notes:

1. The geodesic eqn describes all types of geodesics, not just timelike.
2. A geodesic is fully determined by an initial point and direction.

3. Christoffel symbols are not tensors!

4. Note symmetry: $\Gamma_{ijk} = \Gamma_{ikj} \quad ; \quad \Gamma^i_{jk} = \Gamma^i_{kj}$

5. $g_{ij,k} = \Gamma_{ijk} + \Gamma_{jik}$

$\boxed{\text{SP 8.3}}$

In a geodesic coord system (i.e., a LIF), the Christoffel symbols all vanish

=> $\ddot{x}^i = 0$ (which we recognize as the eqn of motion of a free particle in an IF; parameter = τ)

Suppose $\{x^{i'}\}$ is a geodesic coord system and $\{x^i\}$ is an arbitrary coord system

$$\dot{x}^{i'} = \frac{dx^{i'}}{ds} = \frac{\partial x^{i'}}{\partial x^j}\frac{dx^j}{ds} = p^{i'}_j \, \dot{x}^j$$

$$\ddot{x}^{i'} = \frac{d\dot{x}^{i'}}{ds} = \frac{d(p^{i'}_j\,\dot{x}^j)}{ds} = p^{i'}_j\,\ddot{x}^j + \frac{\partial p^{i'}_j}{\partial x^k}\frac{dx^k}{ds}\,\dot{x}^j$$

$$= p^{i'}_i\,\ddot{x}^i + p^{i'}_{jk}\,\dot{x}^j\,\dot{x}^k = 0$$

$$\Rightarrow p^l_{i'}\,p^{i'}_i\,\ddot{x}^i + p^{i'}_{jk}\,p^l_{i'}\,\dot{x}^j\,\dot{x}^k = 0$$

$$\ddot{x}^l + p^{i'}_{jk}\,p^l_{i'}\,\dot{x}^j\,\dot{x}^k = 0 \qquad \text{, or,} \qquad \ddot{x}^i + p^{i'}_{jk}\,p^i_{i'}\,\dot{x}^j\,\dot{x}^k = 0$$

=> $\Gamma^i_{\ jk} = p^{i'}_{jk}\,p^i_{i'}$ (with $\{x^{i'}\}$ a geodesic coord system)

Note: $p^i_{i'}\,p^{i'}_j = \delta^i_{\ j}$ Differentiate wrt x^k : $p^i_{i'}\,p^{i'}_{jk} + p^i_{i'k'}\,p^{k'}_k\,p^{i'}_j = 0$

$$\Rightarrow \Gamma^i_{\ jk} = p^i_{i'}\,p^{i'}_{jk} = -p^i_{i'k'}\,p^{k'}_k\,p^{i'}_j$$

$$= -p^i_{j'k'}\,p^{j'}_j\,p^{k'}_k$$

Recall that partial differentiation is not a tensor operation for non-linear transformations, since derivatives of the *p*'s arise.

Another way to think of this: We have to compare vectors at different points to form a derivative, but we have no way to compare vector directions at different points in a curved spacetime!

To construct a tensorial definition of differentiation: take the derivative in a (locally flat) geodesic coord system and then transform the result into the arbitrary coord system of interest.

Suppose $\{x^{i'}\}$ is some particular geodesic coord system at point P and $\{x^{i}\}$ is an arbitrary coord system.

Define the covariant derivative of a contravariant vector F^{i} as:

$$F^{i}{}_{;j} \equiv F^{i'}{}_{,j'}\, p^{i}_{i'}\, p^{j'}_{j}$$

$$F^{i}{}_{;j} \equiv F^{i'}{}_{,j'}\, p^{i}_{i'}\, p^{j'}_{j} \qquad \Rightarrow \qquad F^{i'}{}_{;j'} = F^{i'}{}_{,j'}$$

By construction, $F^{i}{}_{;j}$ transforms tensorially from $\{x^{i}\}$ to $\{x^{i'}\}$

Now transform from $\{x^{i'}\}$ to another arbitrary system $\{x^{i''}\}$:

$$F^{i''}{}_{;j''} = F^{i'}{}_{,j'}\, p^{i''}_{i'}\, p^{j'}_{j''}$$

This yields the definition of the covariant derivative in $\{x^{i''}\}$

=> it transforms tensorially from $\{x^{i'}\} \rightarrow \{x^{i''}\}$ as well.

Transitivity: if $F^{i}{}_{;j}$ transforms tensorially from $\{x^{i}\} \rightarrow \{x^{i'}\}$

and from $\{x^{i'}\} \rightarrow \{x^{i''}\}$, then it transforms tensorially from

$\{x^{i}\} \rightarrow \{x^{i''}\}$. Since these are both arbitrary coord systems,

$F^{i}{}_{;j}$ is a tensor.

Proof of transitivity:

$$B^{i''}{}_{j''} = B^{i'}{}_{j'}\, p^{i''}_{i'}\, p^{j'}_{j''} = \left(B^{i}{}_{j}\, p^{i'}_{i}\, p^{j}_{j'}\right) p^{i''}_{i'}\, p^{j'}_{j''} = B^{i}{}_{j}\, p^{i''}_{i}\, p^{j}_{j''}$$

Now we seek a formula for $F^{i}{}_{;j}$ that can be expressed entirely within $\{x^{i}\}$

Start with $F^{i'} = F^{i}\, p^{i'}_{i}$ Now differentiate wrt $x^{j'}$:

$$F^{i'}{}_{,j'} = F^{i}{}_{,j}\, p^{j}_{j'}\, p^{i'}_{i} + F^{i}\, p^{i'}_{ij}\, p^{j}_{j'}$$ Dummy index in 2nd term: $i \rightarrow a$:

$$= F^{i}{}_{,j}\, p^{j}_{j'}\, p^{i'}_{i} + F^{a}\, p^{i'}_{aj}\, p^{j}_{j'}$$ Dummy indices: $i \rightarrow k$, $j \rightarrow l$:

$$= F^{k}{}_{,l}\, p^{l}_{j'}\, p^{i'}_{k} + F^{a}\, p^{i'}_{al}\, p^{l}_{j'}$$ Multiply by $p^{i}_{i'}\, p^{j'}_{j}$:

$$F^{i'}{}_{,j'}\, p^{i}_{i'}\, p^{j'}_{j} = F^{k}{}_{,l}\, p^{l}_{j'}\, p^{j'}_{j}\, p^{i'}_{k} p^{i}_{i'} + F^{a}\, p^{i'}_{al}\, p^{i}_{i'}\, p^{l}_{j'}\, p^{j'}_{j}$$

From previous slide:

$$F^{i'}{}_{,j'}\, p^i_{i'}\, p^{j'}_j = F^k{}_{,l}\, p^l_{j'}\, p^{j'}_j\, p^{i'}_k p^i_{i'} + F^a\, p^{i'}_{al}\, p^i_{i'}\, p^l_{j'}\, p^{j'}_j$$

$$F^i{}_{;j} = F^k{}_{,l}\, p^l_j\, p^i_k + F^a\, p^{i'}_{al}\, p^i_{i'}\, p^l_j$$

(Recall that primed coord system is a geodesic coord system.)

$$= F^i{}_{,j} + F^a\, p^{i'}_{aj}\, p^i_{i'}$$

(Recall that $p^l_j = \delta^l{}_j$)

$$\boxed{F^i{}_{;j} = F^i{}_{,j} + F^a\, \Gamma^i_{aj}}$$

So, we can compute covariant derivatives if we know the Christoffel symbols.

SP 8.4

Another way to think about covariant differentiation:

Christoffel symbols account for changes in the basis vectors $\mathbf{e}_i$ as we move from one point to another.

With $\dfrac{\partial \mathbf{e}_i}{\partial x^j} = \Gamma^a_{ij}\,\mathbf{e}_a$, $\dfrac{\partial (F^i \mathbf{e}_i)}{\partial x^j} = \dfrac{\partial F^i}{\partial x^j}\,\mathbf{e}_i + F^i\,\dfrac{\partial \mathbf{e}_i}{\partial x^j}$

$$= F^i{}_{,j}\,\mathbf{e}_i + F^i\,\Gamma^a_{ij}\,\mathbf{e}_a$$

$$= F^i{}_{,j}\,\mathbf{e}_i + F^a\,\Gamma^i_{aj}\,\mathbf{e}_i$$

$$= \left(F^i{}_{,j} + F^a\,\Gamma^i_{aj}\right)\,\mathbf{e}_i$$

$$= F^i{}_{;j}\,\mathbf{e}_i$$

Covariant differentiation of covariant vectors:

Define $F_{i;j} = F_{i',j'}\, p_i^{i'}\, p_j^{j'}$ (primed coord system is a geodesic coord system)

$F_{i'} = F_i\, p_{i'}^{i}$ Differentiate wrt $x^{j'}$:

$$F_{i',j'} = F_{i,j}\, p_{j'}^{j}\, p_{i'}^{i} + F_i\, p_{i'j'}^{i}$$

$$= F_{k,l}\, p_{j'}^{l}\, p_{i'}^{k} + F_a\, p_{i'j'}^{a}$$ Multiply by $p_i^{i'}\, p_j^{j'}$

$$F_{i',j'}\, p_i^{i'}\, p_j^{j'} = F_{k,l}\, p_{j'}^{l}\, p_j^{j'}\, p_{i'}^{k}\, p_i^{i'} + F_a\, p_{i'j'}^{a}\, p_i^{i'}\, p_j^{j'}$$

$$\boxed{F_{i;j} = F_{i,j} - F_a\, \Gamma_{ij}^{a}}$$

For higher rank tensors, follow the pattern for contravariant and covariant vectors.

e.g. $F^i{}_{j;k} = F^i{}_{j,k} + F^a{}_j \Gamma^i_{ak} - F^i{}_a \Gamma^a_{jk}$

$$F^{ij}{}_{kl;m} = F^{ij}{}_{kl,m} + F^{aj}{}_{kl} \Gamma^i_{am} + F^{ia}{}_{kl} \Gamma^j_{am} - F^{ij}{}_{al} \Gamma^a_{km} - F^{ij}{}_{ka} \Gamma^a_{lm}$$

Notes:

1. For a scalar $\phi(x^i)$: $\phi_{;i} = \phi_{,i}$ (no p's in transforming a scalar)

2. Covariant differentiation satisfies the sum and product rules.

e.g.: in geodesic coords, $A^i{}_{,j} + B^i{}_{,j} = (A+B)^i{}_{,j}$

$$\Rightarrow A^i{}_{;j} + B^i{}_{;j} = (A+B)^i{}_{;j}$$

This is a tensor eqn, so it's true in all coord systems.

3. $g_{ij;k} = 0$ and $g^{ij}{}_{;k} = 0$ (since true in geodesic coords)

Then $\delta^i{}_j = g^{ik} g_{kj} \Rightarrow \delta^i{}_{j;k} = 0$

4. Covariant diff commutes with raising/lowering and contraction.

e.g.: $F_{ij;k} = \left(g_{im} F^m{}_j\right)_{;k}$

$$= \underbrace{g_{im;k}}_{0} F^m{}_j + g_{im} F^m{}_{j;k}$$

5. Covariant diff is not commutative!

$F^i{}_{j;kl} \neq F^i{}_{j;lk}$ (more on this later...)

Consider differentiation of a tensor ${F^i}_j$ along a curve $x^i(u)$.

$\frac{d{F^i}_j}{du} = {F^i}_{j,k}\,\dot{x}^k$ is NOT a tensor. ($\dot{x}^k$ is but ${F^i}_{j,k}$ is not.)

Define the absolute derivative (which is a tensor) as:

$$\begin{aligned}\frac{D{F^i}_j}{du} &\equiv {F^i}_{j;k}\,\dot{x}^k \\ &= \frac{d{F^i}_j}{du} + {F^a}_j\,\Gamma^i_{ak}\,\dot{x}^k - {F^i}_a\,\Gamma^a_{jk}\,\dot{x}^k\end{aligned}$$

Same generalization to other ranks as for covariant differentiation.

e.g., for a scalar ϕ: $\frac{D\phi}{du} = \frac{d\phi}{du}$

Absolute derivative satisfies the sum and product rules.

$$\frac{Dg_{ij}}{du} = 0 \qquad \frac{Dg^{ij}}{du} = 0 \qquad \frac{D\delta^i{}_j}{du} = 0$$

For non-null curves, take $u = s$
and define the unit tangent vector as:

$$T^i = \frac{dx^i}{ds} = \dot{x}^i$$

$$\mathbf{T}^2 = g_{ij}\,\frac{dx^i}{ds}\,\frac{dx^j}{ds} = \frac{d\mathbf{s}^2}{ds^2} = \pm 1$$

Define the principal normal as:

$$N^i = \frac{DT^i}{ds} = \frac{d^2x^i}{ds^2} + \Gamma^i_{jk}\,\frac{dx^j}{ds}\,\frac{dx^k}{ds}$$

$$= \ddot{x}^i + \Gamma^i_{jk}\,\dot{x}^j\,\dot{x}^k$$

$= 0$ for geodesics (=> they are the straightest possible curves)

$$\frac{D\mathbf{T}^2}{ds} = \frac{D(\pm 1)}{ds} = 0$$

$$= \frac{D(g_{ij}\,T^i\,T^j)}{ds} = 2\,g_{ij}\,N^i\,T^j = 2\mathbf{N}\cdot\mathbf{T}$$

$\Rightarrow \mathbf{N}\cdot\mathbf{T} = 0$ so **N** is truly a normal.

The magnitude of **N** is known as the curvature κ of a curve:

$$\kappa = \left|g_{ij}\,N^i\,N^j\right|^{1/2}$$

Now specialize to spacetime.
Recall that proper time $\tau = s/c$

Define generalized 4-velocity **U**, 4-acceleration **A**, and proper acceleration α as:

$$U^{\mu} = \frac{dx^{\mu}}{d\tau} = c\,T^{\mu}$$

$$A^{\mu} = \frac{DU^{\mu}}{d\tau} = c^2\,N^{\mu}$$

$$\alpha = \left|g_{\mu\nu}\,A^{\mu}\,A^{\nu}\right|^{1/2} = c^2\,\kappa$$

These are vectors (**U**, **A**) and a scalar (α) for general spacetime coord transformations and reduce to their SR definitions in the LIFs (geodesic coord systems).

As expected, geodesic worldlines have $\mathbf{A} = 0$ and $\alpha = 0$.

A vector V^i is "parallel transported" along a curve $x^i(u)$ if $\frac{DV^i}{du} = 0$

=> a geodesic parallel transports its tangent vector.

Scalar product of 2 parallel-transported vectors is constant:

$$\frac{D}{du}\left(g_{ij}V^iW^j\right) = g_{ij}\frac{DV^i}{du}W^j + g_{ij}V^i\frac{DW^j}{du} + \frac{Dg_{ij}}{du}V^iW^j = 0$$

0 by defn of p.t.

0 always

As a specific case: the magnitude of a vector remains constant on p.t.

=> 1. angle btwn 2 parallel-transported vectors remains constant.

2. vector parallel-transported along a geodesic subtends a constant angle wrt the tangent.

SP 8.5

Consider the Newtonian limit: a particle moving slowly in a weak, stationary gravitational field

"Stationary" => we can take $x^4 = ct$

Geodesic equation: $$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau} = 0$$

$$v \ll c \quad \Rightarrow \quad \frac{1}{c}\frac{dx^i}{d\tau} \ll \frac{dt}{d\tau} \qquad (i = 1, 2, 3)$$

=> retain only α, β = 4 in 2nd term of geodesic eqn

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{44}\, c^2 \left(\frac{dt}{d\tau}\right)^2 = 0$$

$$\Gamma^\mu_{\alpha\beta} = \frac{1}{2} g^{\mu\nu} \left(g_{\nu\alpha,\beta} + g_{\nu\beta,\alpha} - g_{\alpha\beta,\nu}\right)$$

0, since stationary

$$\Rightarrow \Gamma^\mu_{44} = \frac{1}{2} g^{\mu\nu} \left(g_{\nu 4,4} + g_{\nu 4,4} - g_{44,\nu}\right) \qquad \Rightarrow \qquad \Gamma^\mu_{44} = -\frac{1}{2} g^{\mu\nu}\, g_{44,\nu}$$

$$\frac{d^2x^\mu}{d\tau^2} + \Gamma^\mu_{44}\, c^2 \left(\frac{dt}{d\tau}\right)^2 = 0 \qquad \text{with} \qquad \Gamma^\mu_{44} = -\frac{1}{2}\, g^{\mu\nu}\, g_{44,\nu}$$

Weak field => $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ $\qquad |h_{\alpha\beta}| \ll 1$

$$\Rightarrow \Gamma^\mu_{44} \approx -\frac{1}{2}\,\eta^{\mu\nu}\,\frac{\partial h_{44}}{\partial x^\nu} \qquad \Rightarrow \frac{d^2x^\mu}{d\tau^2} - \frac{c^2}{2}\,\eta^{\mu\nu}\,\frac{\partial h_{44}}{\partial x^\nu}\left(\frac{dt}{d\tau}\right)^2 = 0$$

$$\eta^{\mu\nu}\,\frac{\partial h_{44}}{\partial x^\nu} = -\frac{\partial h_{44}}{\partial x^\mu}\ , \quad \mu = 1,2,3$$

$$= \frac{\partial h_{44}}{\partial x^4}\ , \quad \mu = 4$$

=> $\displaystyle \frac{d^2\mathbf{x}}{d\tau^2} = -\frac{c^2}{2}\nabla h_{44}\left(\frac{dt}{d\tau}\right)^2$ $\qquad \displaystyle \frac{d^2t}{d\tau^2} = \frac{1}{2}\left(\frac{dt}{d\tau}\right)^2\frac{\partial h_{44}}{\partial t} = 0$

=> $\displaystyle \frac{dt}{d\tau} = \text{const}$

$$\frac{d^2\mathbf{x}}{d\tau^2} = -\frac{c^2}{2}\nabla h_{44}\left(\frac{dt}{d\tau}\right)^2 \qquad \text{and} \qquad \frac{dt}{d\tau} = \text{const}$$

=> $$\frac{d^2\mathbf{x}}{dt^2} = -\frac{c^2}{2}\nabla h_{44}$$ c.f. Newtonian result: $$\frac{d^2\mathbf{x}}{dt^2} = -\nabla\phi$$

$$\Rightarrow \phi = \frac{c^2}{2}h_{44} + \text{const}$$

$h_{44}\ ,\ \phi \to 0$ at ∞ $\Rightarrow \text{const} = 0$

=> $$h_{44} = \frac{2\phi}{c^2}$$

=> weak grav field is characterized by $g_{44} = 1 + \dfrac{2\phi}{c^2}$

=> constraint on developing Einstein's eqns (soon to come)