Curvature and Einstein's Field Equations

Goal: Relate the curvature of spacetime to the sources

Newtonian theory: mass density is the source

SR: $E = mc^2$ => energy density should be involved

Consider dust = a collection of non-interacting particles, all doing the same thing (simplest continuous source)

In rest frame: energy density = $m_0 c^2 n_0$ (m_0 is rest mass and n_0 is proper number density)

In an arbitrary frame: energy density increases by factor γ^2, since $E = \gamma m_0 c^2$ and $n = \gamma n_0$ (because of length contraction).

If energy density were a component of a 4-vector, it would only pick up one factor of γ => we should look for an appropriate rank 2 tensor.

Stress-energy tensor (or, energy-momentum tensor, ...):

$T^{\mu\nu}$ = c × the "flux" of μ-component of 4-momentum crossing a surface of constant x^ν. "Flux" means $dP^\mu/(dx^\alpha dx^\beta dx^\gamma)$, where α, β, γ are the 3 index values other than ν.

Consider dust moving at constant speed along x-axis:

$T^{44} = c \times E/c$ per unit volume crossing a surface of constant ct

$= \text{energy density} = \gamma^2 m_0 c^2 n_0 = m_0 n_0 U^4 U^4$

$T^{xx} = c \times p^x$ per unit area per unit ct = momentum per (area time)

$= p^x \text{ per volume} \times v^x = \gamma^2 m_0 n_0 (v^x)^2 = m_0 n_0 U^x U^x$

Generally, for dust: $T^{\mu\nu} = m_0 n_0 U^\mu U^\nu$ (symmetric)

For general source (more complicated fluids, em fields, etc.), $T^{\mu\nu}$ remains symmetric.

Recall the continuity eqn in electrodynamics: $J^{\mu}{}_{,\mu} = 0$ (conservation of electric charge).

Conservation of energy and momentum is expressed by $T^{\mu\nu}{}_{,\nu} = 0$
(stress-energy tensor is divergenceless)

Recall the strong equivalence principle: the laws of physics in a LIF are the same as in SR.

Laws are expressed as tensor equations.

All of the SR tensor operations remain valid in GR except differentiation.

Covariant differentiation is a tensor operation for arbitrary transformations in curved spacetime and is equivalent to partial differentiation in geodesic coord systems (LIFs).

To get a law of physics in GR, take the law in SR and change the partial derivatives to covariant derivatives ("comma-goes-to-semicolon rule").

=> $T^{\mu\nu}{}_{;\nu} = 0$

Curvature:

All info on the intrinsic curvature of a space is contained within g_{ij} .

However: it is generally very difficult to extract this info (metric tensor also depends on the coord system you use).

For example: $d\mathbf{s}^2 = (du^2 + dv^2)\exp(u - v)$ flat

$d\mathbf{s}^2 = dw^2 + d\phi^2 \exp(-2w)$ curved

Thus, we seek a tensorial description of intrinsic curvature (i.e., a tensor derived from g_{ij} that tells us immediately about the curvature).

Should depend on 2nd derivatives of g_{ij} (local flatness theorem).

Definition of Riemann curvature tensor ${R^h}_{ajk}$:

$${V^h}_{;jk} - {V^h}_{;kj} = -V^a\,{R^h}_{ajk}$$ (true for arbitrary **V** => tensor)

$$\Rightarrow {R^h}_{ijk} = \Gamma^h_{ik,j} - \Gamma^h_{ij,k} + \Gamma^h_{aj}\,\Gamma^a_{ik} - \Gamma^h_{ak}\,\Gamma^a_{ij}$$

$$R^h{}_{ijk} = \Gamma^h_{ik,j} - \Gamma^h_{ij,k} + \Gamma^h_{aj}\,\Gamma^a_{ik} - \Gamma^h_{ak}\,\Gamma^a_{ij}$$

$R^h{}_{ijk} = 0$ in flat space, since it vanishes in Cartesian coords and is a tensor.

Conversely, its global vanishing implies that the space is flat.

In geodesic coords, Christoffel symbols vanish (but their derivatives don't)

$$\Rightarrow \quad R^h{}_{ijk} = \Gamma^h_{ik,j} - \Gamma^h_{ij,k}$$

$$= \frac{1}{2} g^{ah} \left(\cancel{g_{ai,kj}} + g_{ak,ij} - g_{ik,aj} - \cancel{g_{ai,jk}} - g_{aj,ik} + g_{ij,ak} \right)$$

$$R_{lijk} = g_{lh}\, R^h{}_{ijk} = \frac{1}{2} g_{lh}\, g^{ha} (\ldots)$$

$$R_{lijk} = \frac{1}{2} \left(g_{lk,ij} + g_{ij,lk} - g_{ik,lj} - g_{lj,ik} \right)$$

$$R_{hijk} = \frac{1}{2}\left(g_{hk,ij} + g_{ij,hk} - g_{ik,hj} - g_{hj,ik}\right)$$

=> $R_{hijk} = -R_{hikj}$

$$R_{hijk} = -R_{ihjk}$$

$$R_{hijk} = R_{jkhi}$$

$R_{hijk} + R_{hjki} + R_{hkij} = 0$ "cyclic identity"

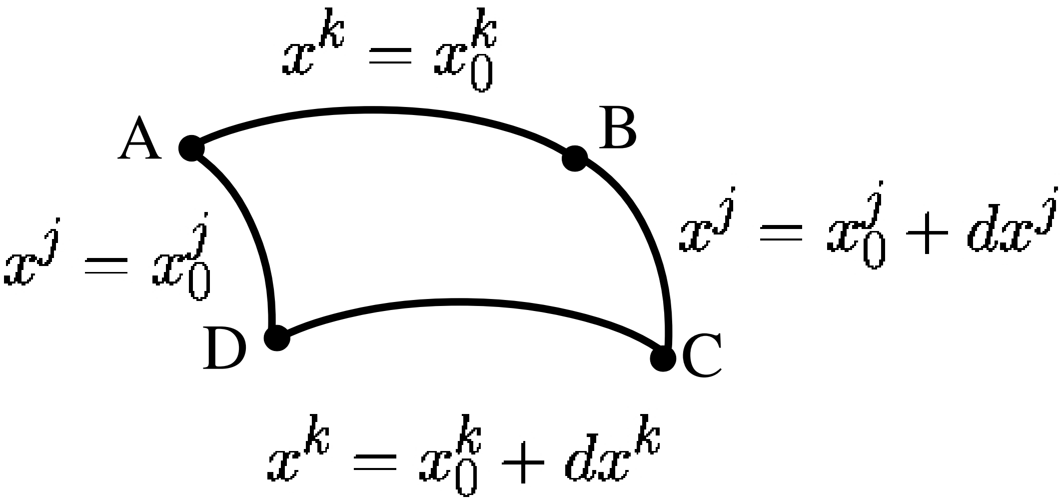
Symmetries reduce the number of independent components of R_{hijk} from N^4 to $N^2(N^2\text{-}1)/12$

=> 20 for $N = 4$ (as expected)

=> 1 for $N = 2$ (all non-zero components = $\pm R_{1212}$)

$\boxed{\text{SP 9.1-2}}$

Relation of Riemann tensor to parallel transport:



Consider parallel transport around a small “parallelogram” in the x^j, x^k “plane”.

Parallel transport: $$\frac{DV^h}{du} = 0 \quad \Rightarrow \quad \frac{dV^h}{du} + V^i \Gamma^h_{im} \dot{x}^m = 0$$

$$\mathrm{A} \rightarrow \mathrm{B}: \; u = x^j \quad \Rightarrow \quad \frac{\partial V^h}{\partial x^j} = -V^i \, \Gamma^h_{ij}$$

($\dot{x}^m = 1$ when $m=j$ and 0 otherwise)

$$\Rightarrow \; dV^h = -V^i \, \Gamma^h_{ij} \, dx^j$$

$$\Rightarrow \; V^h(\mathrm{B}) - V^h(\mathrm{A}) = - \int_{x_0^j}^{x_0^j + dx^j} \left(V^i \Gamma^h_{ij} \right)_{x^k = x_0^k} dx^j$$

From previous slide: $$V^h(\mathrm{B}) - V^h(\mathrm{A}) = -\int_{x_0^j}^{x_0^j+dx^j} \left(V^i\Gamma^h_{ij}\right)_{x^k=x_0^k} dx^j$$

Similarly:

$$V^h(\mathrm{C}) - V^h(\mathrm{B}) = -\int_{x_0^k}^{x_0^k+dx^k} \left(V^i\Gamma^h_{ik}\right)_{x^j=x_0^j+dx^j} dx^k$$

(take $u = x^k$ for this leg)

$$V^h(\mathrm{D}) - V^h(\mathrm{C}) = \int_{x_0^j}^{x_0^j+dx^j} \left(V^i\Gamma^h_{ij}\right)_{x^k=x_0^k+dx^k} dx^j$$

(no minus sign since we're traversing in the opposite dir)

$$V^h(\mathrm{A,end}) - V^h(\mathrm{D}) = \int_{x_0^k}^{x_0^k+dx^k} \left(V^i\Gamma^h_{ik}\right)_{x^j=x_0^j} dx^k$$

$$dV^h = V^h(\mathrm{A,end}) - V^h(\mathrm{A})$$

$$dV^h = \int_{x_0^j}^{x_0^j+dx^j} dx^j \left[\left(V^i\,\Gamma^h_{ij}\right)_{x_0^k+dx^k} - \left(V^i\,\Gamma^h_{ij}\right)_{x_0^k}\right]$$

$$- \int_{x_0^k}^{x_0^k+dx^k} dx^k \left[\left(V^i\,\Gamma^h_{ik}\right)_{x_0^j+dx^j} - \left(V^i\,\Gamma^h_{ik}\right)_{x_0^j}\right]$$

Expand to 1st order: $\left(V^i\,\Gamma^h_{ij}\right)_{x_0^k+dx^k} - \left(V^i\,\Gamma^h_{ij}\right)_{x_0^k} = \frac{\partial}{\partial x^k}\left(V^i\,\Gamma^h_{ij}\right)\,dx^k$

=> $dV^h = dx^j\,dx^k\left[\frac{\partial}{\partial x^k}\left(V^i\,\Gamma^h_{ij}\right) - \frac{\partial}{\partial x^j}\left(V^i\,\Gamma^h_{ik}\right)\right]$

$$= dx^j\,dx^k\left[V^i{}_{,k}\,\Gamma^h_{ij} + V^i\,\Gamma^h_{ij,k} - V^i{}_{,j}\,\Gamma^h_{ik} - V^i\,\Gamma^h_{ik,j}\right]$$

Note: $V^i{}_{,k} = -V^a\,\Gamma^i_{ak}$ and $V^i{}_{,j} = -V^a\,\Gamma^i_{aj}$ (parallel transport; absolute deriv = 0)

$$dV^h = dx^j\,dx^k\left[-V^a\,\Gamma^i_{ak}\,\Gamma^h_{ij} + V^i\,\Gamma^h_{ij,k} + V^a\,\Gamma^i_{aj}\,\Gamma^h_{ik} - V^i\,\Gamma^h_{ik,j}\right]$$

$$= dx^j\,dx^k\left[-V^i\,\Gamma^a_{ik}\,\Gamma^h_{aj} + V^i\,\Gamma^h_{ij,k} + V^i\,\Gamma^a_{ij}\,\Gamma^h_{ak} - V^i\,\Gamma^h_{ik,j}\right]$$

$$= -dx^j\,dx^k\,V^i\,\left[\Gamma^h_{ik,j} - \Gamma^h_{ij,k} + \Gamma^h_{aj}\,\Gamma^a_{ik} - \Gamma^h_{ak}\,\Gamma^a_{ij}\right]$$

=> $dV^h = -R^h{}_{ijk}\,V^i\,dx^j\,dx^k$

=> Change in vector on parallel transport
around a small closed loop $\propto R^h{}_{ijk}$

Geodesic deviation:

Consider 2 neighboring geodesics, A and B, parameterized by *s* of A.

Define $\eta^h(s) = x^h_B(s) - x^h_A(s)$

$$\frac{D^2\eta^h}{ds^2} = R^h{}_{ijk}\,\dot{x}^i\,\dot{x}^j\,\eta^k$$

SP 9.3

=> the relative acceleration of 2 freely-falling particles is proportional to the Riemann tensor.

Relative acceleration is due to the tidal field, which is due to the curvature of spacetime and is described by the Riemann tensor. There is no tensor that describes the gravitational field itself, since this can always be transformed away by going to a LIF.

We seek to construct, from the Riemann tensor, a symmetric, divergenceless, rank 2 tensor to relate to $T^{\mu\nu}$:

In geodesic coords: $R^h{}_{ijk;l} = \Gamma^h_{ik,jl} - \Gamma^h_{ij,kl}$

=> $R^h{}_{ijk;l} + R^h{}_{ikl;j} + R^h{}_{ilj;k} = 0$ (the Bianchi identity)

Define the Ricci tensor: $R_{ij} = R^h{}_{ijh}$

The Ricci tensor is symmetric:

$$R_{ij} = R^h{}_{ijh} = R_{jh}{}^h{}_i = R_j{}^h{}_{hi} = R^h{}_{jih} = R_{ji} \qquad \text{(Rindler, p. 219)}$$

The other contractions of the Riemann tensor do not yield new tensors:

$$R^h{}_{hjk} = 0$$

$$R^h{}_{ihj} = -R^h{}_{ijh} = -R_{ij}$$

Next, define the curvature scalar: $R = R^i{}_i = R^{ij}{}_{ji}$

In the Bianchi identity, contract *h* with *k*:

$$R^h{}_{ijh;l} + R^h{}_{ihl;j} + R^h{}_{ilj;h} = 0$$

$$R_{ij;l} - R_{il;j} + R^h{}_{ilj;h} = 0$$

$$R^i{}_{i;l} - R^i{}_{l;i} + R^{hi}{}_{li;h} = 0$$ (raising *i* and contracting it with *j*)

$$R_{;l} - R^i{}_{l;i} + R^{ih}{}_{lh;i} = 0$$ (exchange dummy index pairs in last term)

$$R_{,l} - R^i{}_{l;i} - R^{hi}{}_{lh;i} = 0$$ (make use of asymmetry)

$$R_{,l} - R^i{}_{l;i} - R^i{}_{l;i} = 0 \quad \Rightarrow \quad R_{,l} - 2R^i{}_{l;i} = 0$$

Define the Einstein tensor: $G_{ij} = R_{ij} - \frac{1}{2} g_{ij} R$

$$G^i{}_j = R^i{}_j - \frac{1}{2} g^i{}_j R$$

$$= R^i{}_j - \frac{1}{2} \delta^i{}_j R$$

$$G^i{}_{j;i} = R^i{}_{j;i} - \frac{1}{2} \delta^i{}_j R_{,i} - \frac{1}{2} \delta^i{}_{j;i} R$$

$$= R^i{}_{j;i} - \frac{1}{2} R_{,j} = 0$$ (from last equation on previous slide)

=> Einstein tensor is symmetric and divergenceless!

Einstein's field equations: $G_{\mu\nu} = -\kappa\, T_{\mu\nu}$ (κ is a constant to be determined)

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\kappa\, T_{\mu\nu}$$

Raise μ and contract with ν:

$$R - \frac{1}{2} \delta^{\mu}{}_{\mu} R = -\kappa T$$

($T = T^{\mu}{}_{\mu}$, the trace of $T^{\mu\nu}$)

=> $R = \kappa T$

Substitute for *R* above:

$$R_{\mu\nu} - \frac{1}{2} \kappa\, T\, g_{\mu\nu} = -\kappa\, T_{\mu\nu} \quad \Rightarrow \quad R_{\mu\nu} = -\kappa \left(T_{\mu\nu} - \frac{1}{2} T\, g_{\mu\nu} \right)$$

When $T_{\mu\nu} = 0$, $R_{\mu\nu} = 0$ ("vacuum field equations"; requires no particles and no electromagnetic fields)

To find κ, consider dust at rest in the stationary, weak field limit

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad , \quad |h_{\mu\nu}| \ll 1$$

$$T_{\mu\nu} = \rho U_\mu U_\nu \qquad (\rho = m_0 n_0)$$

Only $T_{44} \approx \rho c^2$ is non-vanishing.

$$T = T^\mu{}_\mu = g_{44} T^{44} \approx T^{44} \qquad (\text{since } g_{44} \approx 1)$$

$$R_{44} = -\kappa \left(T_{44} - \frac{1}{2} T_{44} g_{44} \right) \approx -\frac{1}{2}\kappa T_{44} = -\frac{1}{2}\rho\kappa c^2$$

$$R_{44} = R^\alpha{}_{44\alpha}$$

$$= \Gamma^\alpha_{4\alpha,4} - \Gamma^\alpha_{44,\alpha} + \Gamma^\alpha_{\beta 4}\Gamma^\beta_{4\alpha} - \Gamma^\alpha_{\beta\alpha}\Gamma^\beta_{44}$$

0 (stationary)

2nd order in $h_{\mu\nu}$

$$R_{44} = -\Gamma^{\alpha}_{44,\alpha} = -\Gamma^{i}_{44,i}$$

Recall $$\Gamma^{i}_{44} = \frac{1}{2}\, h_{44,i} = \frac{1}{2}\, \left(\frac{2\phi}{c^2}\right)_{,i} = \frac{1}{c^2}\,\phi_{,i}$$

=> $$R_{44} = -\frac{1}{c^2}\,\nabla^2\phi = -\frac{4\pi G\rho}{c^2}$$

=> $$-\frac{1}{2}\rho\kappa c^2 = -\frac{4\pi G\rho}{c^2}$$

=> $$\kappa = \frac{8\pi G}{c^4}$$

Einstein's equations $R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = -\frac{8\pi G}{c^4} T_{\mu\nu}$

are 10 equations (since the tensors are symmetric)

They are differential equations, 2nd order in $g_{\mu\nu}$, and very non-linear.
(The Riemann tensor involves derivatives and products of Christoffel symbols, which involve 1. derivatives of $g_{\mu\nu}$ and 2. $g^{\mu\nu}$, which is the inverse of $g_{\mu\nu}$)

Non-linearity => the sum of separate solutions is not a solution; implicitly accounts for the fact that gravity gravitates (i.e., the energy in spacetime curvature contributes to spacetime curvature). There would be no way to include this term explicitly in the field equation, because any "gravity tensor" could be transformed away by adopting a LIF.

Since a metric is required simply to express $T^{\mu\nu}$ and the stress-energy tensor determines the metric, an iterative solution procedure is usually needed (except if symmetries can be exploited).

Einstein's equations can be regarded as consistency conditions that must be satisfied by the energy and spacetime geometry jointly.

Remarkably, the field equations imply the geodesic law of motion (it is not required as an additional axiom); other configurations do not satisfy the equations.