Introduction to Electrostatics

Reading: Jackson 1.1 through 1.9, 1.11
Griffiths Ch 1 and Appendix A

Brief review of vector calculus

Consider a scalar field $F(\vec{x})$

The gradient $\nabla F(\vec{x})$ tells us how *F* varies on small displacements $d\vec{x}$:

$$dF = F(\vec{x} + d\vec{x}) - F(\vec{x}) = d\vec{x} \cdot \nabla F(\vec{x})$$

=> ∇F points in dir. of max increase of *F*; $|\nabla F|$ = rate of increase of *F* along this dir.

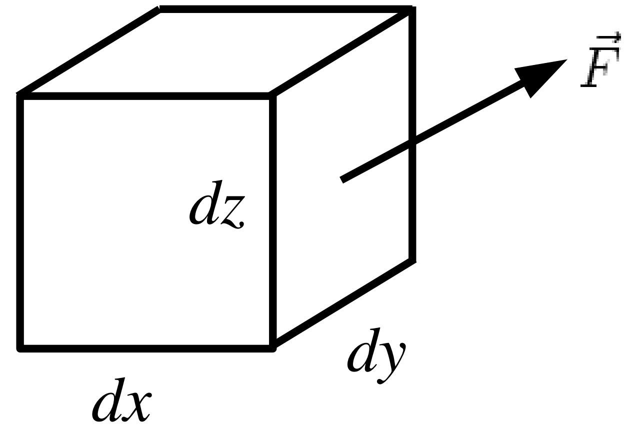
In Cartesian coords: $d\vec{x} \cdot \nabla F = dF = \frac{\partial F}{\partial x}dx + \frac{\partial F}{\partial y}dy + \frac{\partial F}{\partial z}dz$ (using chain rule)

$$\Rightarrow \quad \nabla F = \left(\frac{\partial F}{\partial x}, \frac{\partial F}{\partial y}, \frac{\partial F}{\partial z}\right) = \frac{\partial F}{\partial x}\hat{x} + \frac{\partial F}{\partial y}\hat{y} + \frac{\partial F}{\partial z}\hat{z}$$

Can think of ∇ as a differential operator: $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

Now consider a vector field $\vec{F}(\vec{x})$

The divergence $\nabla \cdot \vec{F}$ is the flux of $\vec{F}$ emerging from an infinitesimal volume, per unit volume:



Flux out of x faces: $$\left[\vec{F}(x+dx,y,z)\cdot\hat{x} - \vec{F}(x,y,z)\cdot\hat{x}\right]\,dy\,dz = \frac{\partial F_x}{\partial x}dV$$

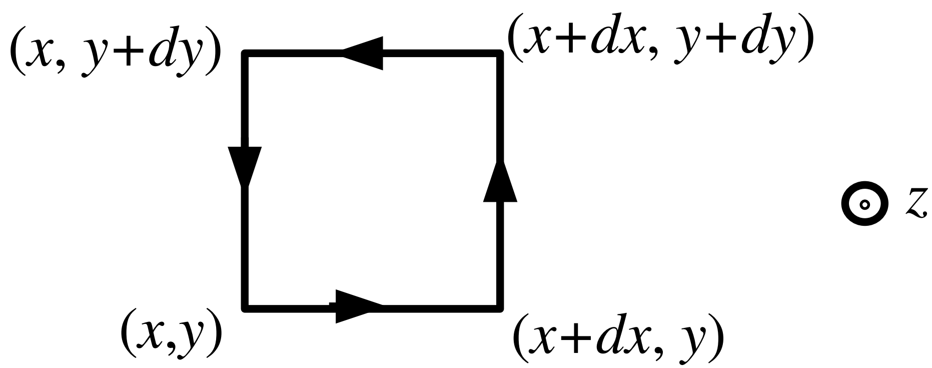
Similarly for y, z => flux out of cube $$= \left(\frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z}\right)dV$$

$$\Rightarrow \quad \nabla\cdot\vec{F} = \frac{\partial F_x}{\partial x} + \frac{\partial F_y}{\partial y} + \frac{\partial F_z}{\partial z} \quad \text{in Cartesian coords} \quad = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\cdot\vec{F}$$

Dividing a finite region into infinitesimal blocks and noting pair-wise cancellation in interior

=> Divergence Theorem: $\int_V \nabla \cdot \vec{F} dV = \oint_S \vec{F} \cdot d\vec{a}$

The curl $\nabla \times \vec{F}$ tells us about the circulation of the vector field:



The x,y,z-component of $\nabla \times \vec{F}$ is the line integral of the vector field around an infinitesimal closed loop, $\perp$ to x,y,z, per unit area

Line integral:

$$\begin{aligned} \oint \vec{F} \cdot d\vec{l} &= \int_x^{x+dx} \left[F_x(x', y) - F_x(x', y+dy)\right] dx' + \int_y^{y+dy} \left[F_y(x+dx, y') - F_y(x, y')\right] dy' \\ &= \int_x^{x+dx} dx' \int_y^{y+dy} dy' \left(\frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right) \end{aligned}$$

$\Rightarrow \quad \dfrac{\partial F_y}{\partial x} - \dfrac{\partial F_x}{\partial y}$ is the *z*-component of $\nabla \times \vec{F}$

Similarly for *x*, *y*-components

$$\Rightarrow \quad \nabla \times \vec{F} = \left(\frac{\partial F_z}{\partial y} - \frac{\partial F_y}{\partial z}, \frac{\partial F_x}{\partial z} - \frac{\partial F_z}{\partial x}, \frac{\partial F_y}{\partial x} - \frac{\partial F_x}{\partial y} \right)$$
$$= \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) \times \vec{F}$$

Dividing a finite surface into infinitesimal squares and noting pair-wise cancellation in interior

=> Stokes's Theorem: $\int (\nabla \times \vec{F}) \cdot d\vec{a} = \oint \vec{F} \cdot d\vec{l}$

Now consider curvilinear coord systems (still in 3D Euclidean space, orthonormal at each point).

Line element $d\vec{l} = f\, du\, \hat{u} + g\, dv\, \hat{v} + h\, dw\, \hat{w}$ (*f*, *g*, *h* are called scale factors)

(For those who took relativity: *f*, *g*, *h* = $\sqrt{g_{uu}}, \sqrt{g_{vv}}, \sqrt{g_{ww}}$)

For example, spherical coords $(u, v, w) = (r, \theta, \phi)$

$$d\vec{l} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\,\sin\theta\,d\phi\,\hat{\phi} \qquad \Rightarrow \qquad f = 1, \quad g = r, \quad h = r\,\sin\theta$$

Again, for a scalar field

$$\begin{aligned} dF &= d\vec{x} \cdot \nabla F \\ &= (\nabla F)_u\, f\, du + (\nabla F)_v\, g\, dv + (\nabla F)_w\, h\, dw \end{aligned}$$

Also,

$$dF = \frac{\partial F}{\partial u} du + \frac{\partial F}{\partial v} dv + \frac{\partial F}{\partial w} dw$$

$$\Rightarrow \quad \nabla F = \frac{1}{f}\frac{\partial F}{\partial u}\,\hat{u} + \frac{1}{g}\frac{\partial F}{\partial v}\,\hat{v} + \frac{1}{h}\frac{\partial F}{\partial w}\,\hat{w}$$

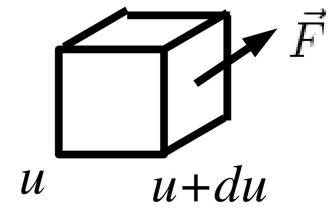
So, for spherical coords ($f = 1$, $g = r$, $h = r \sin\theta$):

$$\nabla F = \frac{\partial F}{\partial r}\,\hat{r} + \frac{1}{r}\frac{\partial F}{\partial \theta}\,\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial F}{\partial \phi}\,\hat{\phi}$$

SP 1.1

Divergence:

Flux out of *u* faces:



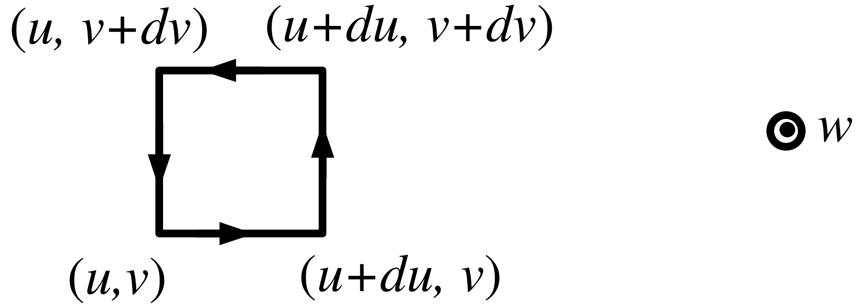
$$F_u(u+du)\,g(u+du)\,h(u+du)\,dv\,dw - F_u(u)\,g(u)\,h(u)\,dv\,dw = \frac{\partial}{\partial u}\left(F_u g h\right)\,du\,dv\,dw$$
$$= \frac{1}{fgh}\frac{\partial}{\partial u}\left(F_u g h\right)\,dV$$

Similarly for *v*, *w* => $\nabla\cdot\vec{F} = \frac{1}{fgh}\left[\frac{\partial}{\partial u}\left(F_u g h\right) + \frac{\partial}{\partial v}\left(F_v f h\right) + \frac{\partial}{\partial w}\left(F_w f g\right)\right]$

So, for spherical coords ($f = 1$, $g = r$, $h = r\sin\theta$):

$$\begin{aligned}\nabla\cdot\vec{F} &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial r}(r^2\sin\theta\,F_r) + \frac{\partial}{\partial\theta}(r\sin\theta\,F_\theta) + \frac{\partial}{\partial\phi}(rF_\phi)\right] \\ &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2F_r) + \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\,F_\theta) + \frac{1}{r\sin\theta}\frac{\partial F_\phi}{\partial\phi}\end{aligned}$$

Curl



$$\oint \vec{F}\cdot d\vec{l} \;=\; \int_u^{u+du} \left[F_u(u',v)f(u',v) - F_u(u',v+dv)f(u',v+dv)\right] du'$$
$$+\int_v^{v+dv} \left[F_v(u+du,v')g(u+du,v') - F_v(u,v')g(u,v')\right] dv'$$
$$= \int_u^{u+du} du' \int_v^{v+dv} dv' \left[\frac{\partial}{\partial u}(F_v g) - \frac{\partial}{\partial v}(F_u f)\right]$$
$$= \int_u^{u+du} f du' \int_v^{v+dv} g dv' \frac{1}{fg}\left[\frac{\partial}{\partial u}(F_v g) - \frac{\partial}{\partial v}(F_u f)\right]$$

=> *w*-component of $\nabla \times \vec{F} = \frac{1}{fg}\left[\frac{\partial}{\partial u}(F_v g) - \frac{\partial}{\partial v}(F_u f)\right]$

Similarly for *u*, *v*-components =>

$$\begin{aligned}\nabla\times\vec{F} &= \frac{1}{gh}\left[\frac{\partial}{\partial v}(F_w h)-\frac{\partial}{\partial w}(F_v g)\right]\hat{u}+\frac{1}{fh}\left[\frac{\partial}{\partial w}(F_u f)-\frac{\partial}{\partial u}(F_w h)\right]\hat{v}\\ &\quad +\frac{1}{fg}\left[\frac{\partial}{\partial u}(F_v g)-\frac{\partial}{\partial v}(F_u f)\right]\hat{w}\end{aligned}$$

So, for spherical coords ($f = 1$, $g = r$, $h = r\sin\theta$):

$$\begin{aligned}\nabla\times\vec{F} &= \frac{1}{r^2\sin\theta}\left[\frac{\partial}{\partial\theta}(r\sin\theta F_\phi)-\frac{\partial}{\partial\phi}(rF_\theta)\right]\hat{r}+\frac{1}{r\sin\theta}\left[\frac{\partial F_r}{\partial\phi}-\frac{\partial}{\partial r}(r\sin\theta F_\phi)\right]\hat{\theta}\\ &\quad +\frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\theta)-\frac{\partial F_r}{\partial\theta}\right]\hat{\phi}\\ &= \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial\theta}(\sin\theta F_\phi)-\frac{\partial F_\theta}{\partial\phi}\right]\hat{r}+\left[\frac{1}{r\sin\theta}\frac{\partial F_r}{\partial\phi}-\frac{1}{r}\frac{\partial}{\partial r}(rF_\phi)\right]\hat{\theta}\\ &\quad +\frac{1}{r}\left[\frac{\partial}{\partial r}(rF_\theta)-\frac{\partial F_r}{\partial\theta}\right]\hat{\phi}\end{aligned}$$

See front and back covers of Jackson for vector identities, theorems, and derivatives in curvilinear coords.

Problem 1: Prove the following variant of Stokes's Theorem:

$$\oint d\vec{l}\,\phi(\vec{x}) = \int da\,\hat{n} \times \nabla\phi(\vec{x})$$

$\hat{n}$ = unit normal vector

Set $\vec{V} = \vec{R}\,\phi$ with $\vec{R}$ a constant vector.

Stokes's Thm: $\oint d\vec{l} \cdot \vec{V} = \int da\,\hat{n} \cdot (\nabla \times \vec{V})$

LHS : $\oint d\vec{l} \cdot \vec{V} = \oint d\vec{l} \cdot (\vec{R}\phi) = \vec{R} \cdot \oint d\vec{l}\,\phi$

$$\begin{aligned}
\text{RHS} : \int da\,\hat{n} \cdot (\nabla \times \vec{V}) &= \int da\,\hat{n} \cdot [\nabla \times (\vec{R}\phi)] \\
&= \int da\,\hat{n} \cdot (\nabla\phi \times \vec{R}) \\
&= \int da\,\vec{R} \cdot (\hat{n} \times \nabla\phi) \\
&= \vec{R} \cdot \int da\,\hat{n} \times \nabla\phi
\end{aligned}$$

$\nabla \times (\vec{R}\phi) = \nabla\phi \times \vec{R} + \phi\,\cancelto{0}{\nabla \times \vec{R}}$

$\vec{A} \cdot (\vec{B} \times \vec{C}) = \vec{C} \cdot (\vec{A} \times \vec{B})$

So : $\vec{R} \cdot \oint d\vec{l}\,\phi = \vec{R} \cdot \int da\,\hat{n} \times \nabla\phi$

Must be true for arbitrary *R* => the integrals are equal.

Now, we focus on electrostatics: $\frac{\partial \rho}{\partial t} = 0 \quad , \quad \frac{\partial \vec{E}}{\partial t} = 0$

Coulomb's Law: $\vec{F} = \frac{1}{4\pi\epsilon_0} q_1 q_2 \frac{\vec{x}_1 - \vec{x}_2}{|\vec{x}_1 - \vec{x}_2|^3}$ (force on 1 by 2)

We adopt SI units; see Jackson appendix for conversion between SI and Gaussian.

Electric field: $\vec{F} = q\vec{E}$

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} q_1 \frac{\vec{x} - \vec{x}_1}{|\vec{x} - \vec{x}_1|^3}$$ (field due to a point charge q_1 at $\vec{x}_1$)

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \rho(\vec{x}') \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} d^3x'$$ (field due to a continuous charge distribution)

Point charges can be treated as a distribution using the Dirac delta function.

In 1D: $\delta(x-a) = 0 \quad \text{for} \quad x \neq a$

$$\int_{x_1}^{x_2} f(x)\delta(x-a)dx = f(a)$$ if the region of integration (x_1, x_2) includes x = a; zero otherwise

Specifically: $\int_{x_1}^{x_2} \delta(x-a)dx = 1$ if region includes x = a

Clearly $\delta(x\text{-}a)$ is undefined at x=a, so it really is not a function.

We can try to express it as a limit, e.g.:

$$\delta_n(x) = \begin{cases} 0 & , x < -1/2n \\ n & , -1/2n < x < 1/2n \\ 0 & , x > 1/2n \end{cases}$$

But $\lim_{n\to\infty} \delta_n(x)$ does not exist.

$$\lim_{n\to\infty} \int_{-\infty}^{\infty} f(x)\delta_n(x)dx = f(0)$$

This limit does exist!

Thus, $\delta(x)$ makes sense when it appears as part of an integrand—this is the only context in which it should be used.

In 3D : $\delta(\vec{x}-\vec{X})=\delta(x_1-X_1)\,\delta(x_2-X_2)\,\delta(x_3-X_3)$

$$\int_{\Delta V} f(\vec{x})\delta(\vec{x}-\vec{X})d^3x=\begin{cases} f(\vec{X}) & \text{, if } \Delta V \text{ contains } \vec{x}=\vec{X} \\ 0 & \text{, otherwise} \end{cases}$$

With $\rho(\vec{x})=\sum_i q_i\,\delta(\vec{x}-\vec{x}_i)$ for a collection of point charges,

$$\begin{aligned}\vec{E}(\vec{x}) &= \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}d^3x' \\ &= \frac{1}{4\pi\epsilon_0}\int \sum_i q_i\delta(\vec{x}'-\vec{x}_i)\frac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3}d^3x' \\ &= \frac{1}{4\pi\epsilon_0}\sum_i q_i\frac{\vec{x}-\vec{x}_i}{|\vec{x}-\vec{x}_i|^3}\end{aligned}$$

Two expressions involving delta functions, $D_1(x)$ and $D_2(x)$, are equal if:

$\int_{-\infty}^{\infty} f(x)D_1(x)dx=\int_{-\infty}^{\infty} f(x)D_2(x)dx$ for all well-behaved functions $f(x)$.

Problem 2: Show that $\delta(kx) = \frac{1}{|k|}\delta(x)$ when k is any non-zero constant.

For arbitrary $f(x)$:

$$\int_{-\infty}^{\infty} f(x)\delta(kx)dx = \begin{cases} \frac{1}{k}\int_{-\infty}^{\infty} f(u/k)\delta(u)du & , k > 0 \\ -\frac{1}{k}\int_{-\infty}^{\infty} f(u/k)\delta(u)du & , k < 0 \end{cases}$$

$$= \frac{1}{|k|}\int_{-\infty}^{\infty} f(u/k)\delta(u)du \quad = \quad \frac{1}{|k|}f(0) \quad = \quad \frac{1}{|k|}\int_{-\infty}^{\infty} f(x)\delta(x)dx$$

Dirac delta function in curvilinear coords:

$$\begin{aligned} \int \delta(\vec{x} - \vec{a})d^3x \;&=\; \int f\, g\, h\, du\, dv\, dw\, \delta(\vec{x} - \vec{a}) \\ &=\; \int du\, dv\, dw\, \delta(x_u - a_u)\, \delta(x_v - a_v)\, \delta(x_w - a_w) \end{aligned}$$

$$\Rightarrow \qquad \delta(\vec{x} - \vec{a}) = \frac{1}{fgh}\delta(x_u - a_u)\, \delta(x_v - a_v)\, \delta(x_w - a_w)$$

assuming $\vec{a}$ is not a degenerate point, i.e., it is not characterized by more than one set of coord values.

Examples of degenerate points: origin in plane polar coords (multiple ϕ)

z-axis in spherical and cylindrical coords (again, multiple ϕ)

At a point with multiple values of coord w :

$$\delta(\vec{x}-\vec{a}) = \frac{1}{fg\int h\,dw}\,\delta(x_u - a_u)\,\delta(x_v - a_v)$$

Examples

Cylindrical coords: point (r', ϕ', z')

Not on z-axis: $\delta(\vec{x}-\vec{x}') = \frac{1}{r}\,\delta(r-r')\,\delta(\phi-\phi')\,\delta(z-z')$

On z-axis: $\delta(\vec{x}-\vec{x}') = \frac{1}{2\pi r}\,\delta(r)\,\delta(z-z')$

Check: $\int dz \int_0^{2\pi} d\phi \int dr\, r\, \frac{1}{2\pi r}\,\delta(r)\,\delta(z-z') = \frac{2\pi}{2\pi} = 1$

Spherical coords: point (r', θ', ϕ')

Not on z-axis: $\delta(\vec{x} - \vec{x}') = \dfrac{1}{r^2 \sin\theta}\, \delta(r - r')\, \delta(\theta - \theta')\, \delta(\phi - \phi')$

using cos θ rather than θ : $\delta(\vec{x} - \vec{x}') = \dfrac{1}{r^2}\, \delta(r - r')\, \delta(\cos\theta - \cos\theta')\, \delta(\phi - \phi')$

On positive z-axis: $\delta(\vec{x} - \vec{x}') = \dfrac{1}{2\pi r^2 \sin\theta}\, \delta(r - r')\, \delta(\theta)$

using cos θ rather than θ : $\delta(\vec{x} - \vec{x}') = \dfrac{1}{2\pi r^2}\, \delta(r - r')\, \delta(\cos\theta - 1)$

The origin (degenerate in both θ and ϕ): $\delta(\vec{x}) = \dfrac{1}{4\pi r^2}\, \delta(r)$

Note: dimension of delta function = inverse of dimension of argument.

Dimension of $\delta(\vec{x})$ is length^{-3}

$\delta(r)$ is length^{-1}

$\delta(\theta)$ is rad^{-1} (dimensionless)

Examples involving delta function in curvilinear coords

Find the charge density $\rho(\vec{x})$ for the following situations:

1) Spherical coords, charge Q uniformly distributed over a spherical shell with radius R:

$$\rho = \sigma\delta(r - R) \quad , \quad \sigma = \frac{Q}{4\pi R^2} \quad \Rightarrow \quad \rho = \frac{Q}{4\pi R^2}\,\delta(r - R)$$

2) Cylindrical coords, charge per unit length λ uniformly distributed over a cylindrical surface of radius b:

$$\rho = \sigma\,\delta(r - b)$$

$$\lambda\,dz = \sigma\,dA = \sigma\,2\pi b\,dz \quad \Rightarrow \quad \sigma = \frac{\lambda}{2\pi b}$$

$$\Rightarrow \rho = \frac{\lambda}{2\pi b}\,\delta(r - b)$$

3) Cylindrical coords, charge Q spread uniformly over a flat circular disk of negligible thickness and radius R:

$$\rho = \sigma\,\delta(z) \quad , \quad \sigma = \frac{Q}{\pi R^2} \qquad \Rightarrow \qquad \rho = \begin{cases} \frac{Q}{\pi R^2}\,\delta(z) & , r \le R \\ 0 & , r > R \end{cases}$$

4) Same as (3), but in spherical coords:

$$\rho = \sigma\,\frac{\delta(\theta - \pi/2)}{r} \quad , \quad \sigma = \frac{Q}{\pi R^2}$$

$$\Rightarrow \qquad \rho = \begin{cases} \frac{Q}{\pi R^2 r}\,\delta(\theta - \pi/2) & , r \le R \\ 0 & , r > R \end{cases}$$

Check: $$\int_0^{2\pi} d\phi \int d\theta \int_0^R dr\, r^2\, \sin\theta\, \frac{Q}{\pi R^2 r}\,\delta(\theta - \pi/2) \;= 2\pi \cdot 1 \cdot \frac{Q}{\pi R^2} \cdot \frac{R^2}{2} = Q$$

Coulomb's Law plus linear superposition of electric fields yields integral form of Gauss's Law:

$$\oint \vec{E} \cdot d\vec{a} = \frac{q}{\epsilon_0}$$

Divergence Thm yields the differential form: $\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$

Coulomb's Law also yields $\nabla \times \vec{E} = 0$

$\nabla \times \nabla\psi = 0$ => existence of scalar potential: $\vec{E} = -\nabla\Phi$ $\boxed{\text{SP 1.2}}$

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} \, d^3x'$$

Work done on moving a charge against the field:

$$W = -q \int_A^B \vec{E} \cdot d\vec{l} = q \int_A^B \nabla\Phi \cdot d\vec{l} = q \int_A^B d\Phi = q\,(\Phi_B - \Phi_A)$$

Potential energy of a charge q = $q\ \Phi$

$$\oint \vec{E} \cdot d\vec{l} = 0$$

Potential energy of a collection of point charges

Start with one point charge q_1, located at $\vec{x}_1$; its potential $\Phi_1 = \frac{1}{4\pi\epsilon_0} \frac{q_1}{|\vec{x} - \vec{x}_1|}$

Bring next charge, q_2, in from infinity to point $\vec{x}_2$.

Work done against field of 1 is $W = q_2\, \Phi_1(\vec{x}_2) = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{x}_2 - \vec{x}_1|}$

Bring in a 3rd charge, q_3, from infinity to $\vec{x}_3$.

Work done = $q_3 \left[\Phi_1(\vec{x}_3) + \Phi_2(\vec{x}_3)\right]$ etc.

=> potential energy = total work done $= \frac{1}{4\pi\epsilon_0} \sum_{i=1}^{n} \sum_{j<i} \frac{q_i\, q_j}{|\vec{x}_i - \vec{x}_j|} = \frac{1}{8\pi\epsilon_0} \sum_{i,j}^{i\neq j} \frac{q_i\, q_j}{|\vec{x}_i - \vec{x}_j|}$

For a continuous charge dist ρ:

$$W = \frac{1}{8\pi\epsilon_0} \int\int \frac{\rho(\vec{x})\, \rho(\vec{x}')}{|\vec{x} - \vec{x}'|}\, d^3x\, d^3x' = \frac{1}{2} \int \rho(\vec{x})\, \Phi(\vec{x})\, d^3x = \frac{\epsilon_0}{2} \int (\nabla \cdot \vec{E})\, \Phi(\vec{x})\, d^3x$$

Vector identity: $\nabla \cdot (\psi\, \vec{a}) = \vec{a} \cdot \nabla\psi + \psi\, \nabla \cdot \vec{a}$

$$\Rightarrow \quad \nabla \cdot (\Phi\, \vec{E}) = \vec{E} \cdot \nabla\Phi + \Phi\, \nabla \cdot \vec{E} = -|\vec{E}|^2 + \Phi\, \nabla \cdot \vec{E}$$

$$\begin{aligned} \Rightarrow \quad W &= \frac{\epsilon_0}{2} \int \left[\nabla \cdot (\Phi \vec{E}) + |\vec{E}|^2\right] d^3x \\ &= \frac{\epsilon_0}{2} \oint \Phi\, \vec{E} \cdot d\vec{a} + \frac{\epsilon_0}{2} \int |\vec{E}|^2\, d^3x \end{aligned}$$

Taking the bounding surface at infinity, the first term vanishes ($E \propto r^{-2}$, $\Phi \propto r^{-1}$).

=> energy density in the field $w = \dfrac{\epsilon_0}{2} |\vec{E}|^2$

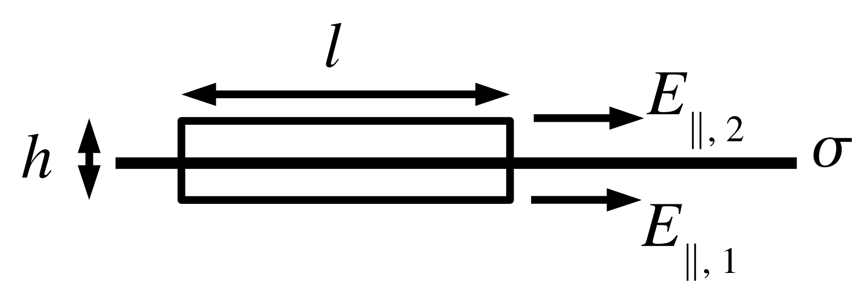
Red flag: Field energy density is non-negative, but energy of a set of point charges can be negative (e.g., 2 charges of opposite sign).
What's going on?

In the integral form of W, $\Phi(\vec{x})$ is the total potential at $\vec{x}$, due to the rest of the charge dist plus the charge at $\vec{x}$. For a truly continuous charge dist, this is the same as the Φ due to the rest of the dist, since the amount of charge at a point vanishes. For a point charge, $q\,\Phi \to \infty$ (and $|\vec{E}|^2 \to \infty$) at the location of the point charge => an infinite "self-energy" that was not included in the energy of the point charge collection (i.e., the charges were taken as already assembled).

Note: infinite self-energy => infinite mass of a point particle unless a negative infinite mass contribution arises from non-electromagnetic source ("renormalization").

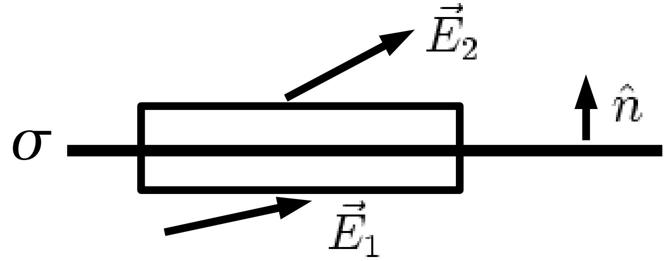
What happens to $\vec{E}$ at a charged surface?

Consider a small rectangular surface $\perp$ to the charged surface:



$\oint \vec{E} \cdot d\vec{l} = (E_{\|,1} - E_{\|,2})\, l$ as $h \to 0$ => tangential component of $\vec{E}$ is continuous across the surface

Gaussian pillbox:



$$\oint \vec{E}\cdot d\vec{a} = (\vec{E}_2 - \vec{E}_1)\cdot\hat{n}\,A = \frac{q}{\epsilon_0} = \frac{\sigma A}{\epsilon_0} \quad \Rightarrow \quad (\vec{E}_2 - \vec{E}_1)\cdot\hat{n} = \frac{\sigma}{\epsilon_0}$$

In a conductor, free charges move in response to an applied $\vec{E}$
Charge flows until $\vec{E} = 0$ inside conductor ($\vec{E}$ of induced charges cancels applied field).

$$\Rightarrow \quad \rho = 0 \ \text{ inside conductor } \quad (\text{since } \nabla\cdot\vec{E} = \nabla\cdot 0 = 0)$$

=> any net and induced charge resides on the surface

Φ = const in a conductor, since $\Delta\Phi = -\int \vec{E}\cdot d\vec{l} = 0$

External $\vec{E}$ is perpendicular to conductor's surface (since tangential component of $\vec{E}$ is continuous across surface and $\vec{E}$ = 0 inside); $\vec{E} = \dfrac{\sigma}{\epsilon_0}\hat{n}$

Suppose we have a conductor held at fixed potential Φ_0. We would like to find Φ everywhere outside the conductor. If we knew how charge distributed itself on the surface of the conductor, we could use

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x' = \frac{1}{4\pi\epsilon_0}\int_S \frac{\sigma(\vec{x}')}{|\vec{x}-\vec{x}'|}\,da'$$

But, can we find $\Phi(\vec{x})$ without knowing $\sigma(\vec{x})$?

Yes! We'll find a differential eqn for $\Phi(\vec{x})$ and apply boundary conditions.

$$\nabla\cdot\vec{E} = \frac{\rho}{\epsilon_0} \quad \text{and} \quad E = -\nabla\Phi$$

$$\Rightarrow \quad \nabla\cdot(-\nabla\Phi) = -\nabla^2\Phi = \frac{\rho}{\epsilon_0} \quad \Rightarrow \quad \nabla^2\Phi = -\frac{\rho}{\epsilon_0} \quad \text{(Poisson Eqn)}$$

For regions where $\rho = 0$, $\quad \nabla^2\Phi = 0 \quad$ (Laplace eqn)

Let's verify that $\Phi_1(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}')}{|\vec{x}-\vec{x}'|}\,d^3x'$ satisfies the Poisson eqn.

$$\nabla^2\Phi_1(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')\,\nabla_x^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)\,d^3x'$$

With $\vec{r} = \vec{x} - \vec{x}'$ and $r = |\vec{r}|$, $\quad \nabla^2\left(\frac{1}{r}\right) = \nabla\cdot\left(\nabla\frac{1}{r}\right) = \nabla\cdot\left(-\frac{\hat{r}}{r^2}\right)$

$$= \frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(-\frac{1}{r^2}\right)\right] = \frac{1}{r^2}\frac{\partial}{\partial r}(-1)$$

= 0 everywhere except r = 0, where r^{-2} is undefined.

$$\begin{aligned}\int \nabla^2\left(\frac{1}{r}\right)\,dV &= \int \nabla\cdot\left(-\frac{\hat{r}}{r^2}\right)\,dV = \oint d\vec{a}\cdot\left(-\frac{\hat{r}}{r^2}\right)\\ &= \int_0^{2\pi} d\phi\int_0^{\pi} d\theta\,\sin\theta\, r^2\,\left(-\frac{1}{r^2}\right)\,(\hat{r}\cdot\hat{r})\\ &= -4\pi\end{aligned}$$

$$\Rightarrow\quad \nabla^2\left(\frac{1}{r}\right) = -4\pi\,\delta(\vec{r}) \quad\Rightarrow\quad \nabla^2\left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = -4\pi\,\delta(\vec{x}-\vec{x}')$$

$$\Rightarrow\quad \nabla^2\Phi_1(\vec{x}) = \frac{1}{4\pi\epsilon_0}\,(-4\pi)\,\rho(\vec{x}) = -\frac{\rho(\vec{x})}{\epsilon_0}$$

Treatment of the boundary value problem is aided by use of Green's Identities.

Start with the Divergence Thm: $\int_V \left(\nabla \cdot \vec{A}\right)\, d^3x = \oint_S \vec{A} \cdot \hat{n}\, da$

Take $\vec{A} = \phi\, \nabla \psi$

$$\nabla \cdot \vec{A} = \nabla \cdot (\phi\, \nabla \psi) = \phi\, \nabla^2 \psi + \nabla \phi \cdot \nabla \psi$$

$\vec{A} \cdot \hat{n} = \phi\, \nabla \psi \cdot \hat{n} = \phi\, \dfrac{\partial \psi}{\partial n}$ ($\partial \psi / \partial n$ = normal derivative on *S*, directed outward from within *V*)

$$\Rightarrow \quad \int_V (\phi\, \nabla^2 \psi + \nabla \phi \cdot \nabla \psi)\, d^3x = \oint_S \phi\, \frac{\partial \psi}{\partial n}\, da$$

(Green's First Identity)

$$\int_V (\psi\, \nabla^2 \phi + \nabla \phi \cdot \nabla \psi)\, d^3x = \oint_S \psi\, \frac{\partial \phi}{\partial n}\, da$$

(interchanging ϕ and ψ)

Subtracting eqns => $\int_V (\phi\, \nabla^2 \psi - \psi\, \nabla^2 \phi)\, d^3x = \oint_S \left[\phi\, \dfrac{\partial \psi}{\partial n} - \psi\, \dfrac{\partial \phi}{\partial n}\right]\, da$

(Green's Theorem)

Take $\phi = \Phi$, $\psi = \dfrac{1}{R} \equiv \dfrac{1}{|\vec{x} - \vec{x}'|}$; $\nabla^2 \Phi = -\dfrac{\rho}{\epsilon_0}$ and $\nabla^2 \left(\dfrac{1}{R}\right) = -4\pi\, \delta(\vec{x} - \vec{x}')$

$$\int_V \left[-4\pi\, \Phi(\vec{x}')\, \delta(\vec{x} - \vec{x}') + \frac{1}{\epsilon_0 R}\, \rho(\vec{x}') \right] d^3x' = \oint_S \left[\Phi \frac{\partial}{\partial n'} \left(\frac{1}{R}\right) - \frac{1}{R} \frac{\partial \Phi}{\partial n'} \right] da'$$

If $\vec{x}$ lies within V :

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V \frac{\rho(\vec{x}')}{R}\, d^3x' + \frac{1}{4\pi} \oint_S \left[\frac{1}{R} \frac{\partial \Phi}{\partial n'} - \Phi \frac{\partial}{\partial n'} \left(\frac{1}{R}\right) \right] da' \qquad (1.36)$$

Note:

1. For surface at ∞, this reduces to our original result for $\Phi(\vec{x})$ (assuming $\vec{E}$ falls off faster than R^{-1}).

2. If $\rho = 0$ in V, Φ anywhere in V depends only on Φ and $\partial\Phi/\partial n$ on S.

Green's 1st Identity leads to powerful conclusions re. the uniqueness of solns to electrostatic boundary value problems. Specifically, problems with Dirichlet or Neumann boundary conditions have unique solns to the Poisson eqn.

Dirichlet: Φ is specified everywhere on the bounding surface.

Neumann: $\partial\Phi/\partial n \, (= -E_n)$ is specified everywhere on bounding surface.

Proof: Suppose there are 2 different solns, Φ_1 and Φ_2

Set $U \equiv \Phi_2 - \Phi_1$

In V: $\nabla^2 U = \nabla^2 \Phi_2 - \nabla^2 \Phi_1 = -\frac{\rho}{\epsilon_0} + \frac{\rho}{\epsilon_0} = 0$

On S: Dirichlet: $U = \Phi_2 - \Phi_1 = 0$

Neumann: $\frac{\partial U}{\partial n} = \frac{\partial \Phi_2}{\partial n} - \frac{\partial \Phi_1}{\partial n} = 0$

Green's 1st Identity with $\phi = \psi = U$:

$$\int_V (U\, \cancel{\nabla^2 U}^{\,0} + \nabla U \cdot \nabla U)\, d^3x = \oint_S U\, \frac{\partial U}{\partial n}\, da$$

product = 0 for both Dirichlet and Neumann

$$\Rightarrow \quad \int_V |\nabla U|^2\, d^3x = 0$$

$$\Rightarrow \quad \nabla U = 0 \quad \Rightarrow \quad U = \text{ const in } V \quad \Rightarrow \quad \Phi_2 = \Phi_1 + \text{ const}$$

=> Φ is unique to within an additive constant.

Suppose the boundary of V consists of the surface of a set of conductors plus a sphere at ∞.

Dirichlet: Φ is specified on each conductor (e.g., by connecting a battery btwn the conductor and ground)

Neumann: $\partial\Phi/\partial n = -E_n = -\sigma/\epsilon_0$ is specified on each conductor.

We don't know σ!

Suppose we know total charge on each conductor. Is soln unique?

In this case, for each conductor surface,

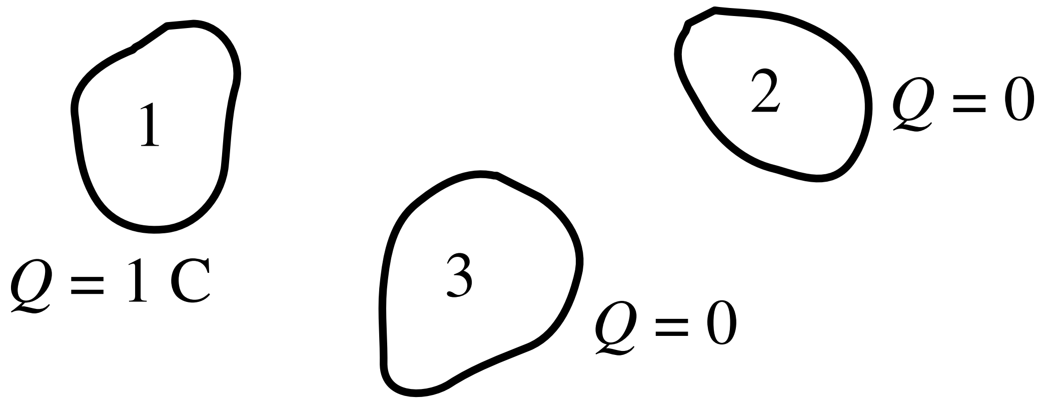
$$
\begin{aligned}
\oint_S U \frac{\partial U}{\partial n}\, da &= \oint_S (\Phi_2 - \Phi_1)\left(\frac{\partial \Phi_2}{\partial n} - \frac{\partial \Phi_1}{\partial n}\right) da \\
&= \oint_S (\Phi_2 - \Phi_1)\,(E_{n,1} - E_{n,2})\, da \\
&= \Phi_2 \oint_S E_{n,1}\, da - \Phi_1 \oint_S E_{n,1}\, da - \Phi_2 \oint_S E_{n,2}\, da + \Phi_1 \oint_S E_{n,2}\, da \\
&= \frac{1}{\epsilon_0}\left(\Phi_2\, Q - \Phi_1\, Q - \Phi_2\, Q + \Phi_1\, Q\right) = 0
\end{aligned}
$$

The potentials can be brought out of the integrals since they are constant on each conductor surface, and the resulting integral = Q by Gauss's Law.

So, again, U = const => Φ is unique to within an additive constant.

Thus, the soln to Poisson's eqn is unique for a region bounded by conductors with either specified potentials or specified charges. (A boundary at infinity is also acceptable.)

Consider a set of conductors. Conductor 1 has $Q = 1$ C and the rest have $Q = 0$.



This has a unique soln for the potential, $\Phi_a(\vec{x})$. The surface charge dist, $\sigma_a(\vec{x})$, is also uniquely determined.

Now alter the charge on conductor 1 from 1 C to α C.

Do the charge dists change?

Suppose the dists are all multiplied by α: $\sigma(\vec{x}) = \alpha\, \sigma_a(\vec{x})$

Q remains 0 on 2 and 3.

Since the Poisson eqn is linear, a new potential $\Phi(\vec{x}) = \alpha\,\Phi_a(\vec{x})$ is a solution.

Since the charge on each conductor is specified, the soln is unique.

So: NO—the charge dists don't change.

If all conductors but conductor 1 have $Q = 0$, then $\Phi \propto Q_1$.

Since the Poisson eqn is linear, the sum of 2 solns is a soln.

In particular, we can add the $\Phi(\vec{x})$ and $\sigma(\vec{x})$ obtained for the case where (1) all but conductor 1 have $Q = 0$ and (2) all but conductor 2 have $Q = 0$.

Again, the soln is unique for these boundary conditions.

$$\Phi(Q_1 = \alpha\,\mathrm{C}, Q_2 = \beta\,\mathrm{C}) = \alpha\,\Phi_a(Q_1 = 1\,\mathrm{C}) + \beta\,\Phi_b(Q_2 = 1\,\mathrm{C})$$

Thus, the potential at the surface of the i^{th} conductor, V_i, is given by

$$V_i = \sum_{j=1}^{n} p_{ij}\,Q_j$$

The p_{ij} are called coefficients of capacity and depend only on the geometry of the conductors.

Inverting the eqns for V_i in terms of Q_j : $\quad Q_i = \sum_{j=1}^{n} C_{ij}\, V_j$

C_{ii} are called capacitances. C_{ij}, $i \neq j$, are called coeffs of induction.

The capacitance C of 2 conductors with $Q_2 = -Q_1$ and potential difference V is defined by $\quad C = \left|\frac{Q_1}{V}\right|$

The potential energy for the system of conductors is

$$
\begin{aligned}
W &= \frac{1}{2}\int \rho(\vec{x})\,\Phi(\vec{x})\,d^3x \\
&= \frac{1}{2}\sum_{i=1}^{n}\int \sigma_i(\vec{x})\,\Phi_i(\vec{x})\,da_i \\
&= \frac{1}{2}\sum_{i=1}^{n} Q_i\,V_i = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} C_{ij}\,V_i\,V_j
\end{aligned}
$$

As a special case, for a capacitor $\quad W = \frac{1}{2}QV = \frac{1}{2}CV^2 = \frac{Q^2}{2C}$

SP 1.3—1.9