# Multipoles, Electrostatics of Macroscopic Media, Dielectrics



Reading: Jackson 4.1 through 4.4, 4.7

Consider a distribution of charge, confined to a region with $r < R$.
Let's expand the resulting potential for $r > R$ in spherical harmonics.
Recall:

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x' \qquad (1)$$

$$\frac{1}{|\vec{x} - \vec{x}'|} = 4\pi \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{1}{2l+1} \frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta', \phi') \, Y_{lm}(\theta, \phi)$$

(Jackson 3.70 and Topic 3, slide 34)

$$r_< = r' \quad \text{and} \quad r_> = r$$

$$\Rightarrow \quad \Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} \left[ \int Y_{lm}^*(\theta', \phi') \, (r')^l \, \rho(\vec{x}') \, d^3x' \right] \frac{Y_{lm}(\theta, \phi)}{r^{l+1}}$$

$$= \frac{1}{4\pi\epsilon_0} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \frac{4\pi}{2l+1} q_{lm} \frac{Y_{lm}(\theta, \phi)}{r^{l+1}}$$

(multipole expansion)

with $q_{lm} = \int Y^*_{lm}(\theta', \phi')\,(r')^l\,\rho(\vec{x}')\,d^3x'$ (multipole moments)

## Why is this useful?

If we calculate $\Phi(\vec{x})$ using (1), then we have to do an integral over $\rho(\vec{x}')$ for each observation point $\vec{x}$. With the multipole expansion, we only have to perform integrals to find the $q_{lm}$. The number of $q_{lm}$ we calculate depends on the required accuracy (potentials from higher multipoles fall off as higher powers of *r*). Once we have the $q_{lm}$, we can find $\Phi(\vec{x})$ for as many $\vec{x}$ as we like without doing any more integrals.

Since $Y_{l,-m}(\theta, \phi) = (-1)^m\,Y^*_{lm}(\theta, \phi)$, (Jackson 3.54 and Topic 3, slide 28)

$$q_{l,-m} = (-1)^m\,q^*_{lm}$$

## Explicit evaluation in Cartesian coords:

$$q_{00} = \int Y^*_{00}(\theta', \phi')\,(r')^0\,\rho(\vec{x}')\,d^3x' = \int \frac{1}{\sqrt{4\pi}}\,\rho(\vec{x}')\,d^3x' = \frac{1}{\sqrt{4\pi}}\,q$$

$$\begin{aligned}
q_{11} &= -\sqrt{\frac{3}{8\pi}}\int \sin(\theta')\,e^{-i\phi'}\,r'\,\rho(\vec{x}')\,d^3x' \\
&= -\sqrt{\frac{3}{8\pi}}\int \sin(\theta')\,(\cos\phi' - i\,\sin\phi')\,r'\,\rho(\vec{x}')\,d^3x' \\
&= -\sqrt{\frac{3}{8\pi}}\int (x' - i\,y')\,\rho(\vec{x}')\,d^3x' \\
&= -\sqrt{\frac{3}{8\pi}}\,(p_x - i\,p_y)
\end{aligned}$$

$$q_{10} = \sqrt{\frac{3}{4\pi}}\int \cos\theta'\,r'\,\rho(\vec{x}')\,d^3x' = \sqrt{\frac{3}{4\pi}}\,p_z$$

where the electric dipole moment $\vec{p} \equiv \int \vec{x}'\,\rho(\vec{x}')\,d^3x'$

The monopole ($l$ = 0) contribution to $\Phi$ is

$$\frac{1}{4\pi\epsilon_0}\frac{4\pi}{2(0)+1}\,\frac{1}{\sqrt{4\pi}}\,q\,\frac{1}{\sqrt{4\pi}}\,\frac{1}{r} = \frac{1}{4\pi\epsilon_0}\frac{q}{r}$$

The dipole ($l$ = 1) contribution to $\Phi$ is

$$\frac{1}{4\pi\epsilon_0}\,\frac{4\pi}{3}\,\frac{1}{r^2}\,\left[q_{11}\,Y_{11}(\theta,\phi)+q_{1,-1}\,Y_{1,-1}(\theta,\phi)+q_{10}\,Y_{10}(\theta,\phi)\right]$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{4\pi}{3}\,\frac{1}{r^2}\,\left[\frac{3}{8\pi}\,(p_x - ip_y)\,\frac{(x+iy)}{r} + \frac{3}{8\pi}\,(p_x+ip_y)\,\frac{(x-iy)}{r} + \frac{3}{4\pi}\,p_z\,\frac{z}{r}\right]$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{1}{r^3}\,\left[\frac{1}{2}\,(p_x x - ip_y x + ip_x y + yp_y + p_x x + ip_y x - ip_x y + yp_y) + p_z z\right]$$

$$= \frac{1}{4\pi\epsilon_0}\,\frac{\vec{p}\cdot\vec{x}}{r^3}$$

Including the monopole, dipole, and quadrupole terms,

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \left[ \frac{q}{r} + \frac{\vec{p}\cdot\vec{x}}{r^3} + \frac{1}{2} \sum_{i,j} Q_{ij} \frac{x_i\, x_j}{r^5} + \dots \right]$$

with $\quad Q_{ij} = \int [3\, x_i'\, x_j' - (r')^2\, \delta_{ij}]\, \rho(\vec{x}')\, d^3x'$

Expansion beyond quadrupole gets messy.

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The electric field due to a dipole at the origin:

$$\vec{E} = -\nabla\Phi = -\frac{1}{4\pi\epsilon_0} \nabla \left( \frac{\vec{p}\cdot\vec{x}}{r^3} \right)$$

$$\nabla \left( \frac{\vec{p}\cdot\vec{x}}{r^3} \right) = \frac{1}{r^3} \nabla(\vec{p}\cdot\vec{x}) + (\vec{p}\cdot\vec{x})\, \nabla \left( \frac{1}{r^3} \right)$$

$$= \frac{1}{r^3} \nabla(p_x x + p_y y + p_z z) + (\vec{p}\cdot\vec{x}) \left( \frac{-3}{r^5} \right) \vec{x} \quad = \frac{\vec{p}}{r^3} + (\vec{p}\cdot\vec{x}) \left( \frac{-3}{r^5} \right) \vec{x}$$

$$\Rightarrow \quad \vec{E} = \frac{3\,(\vec{p}\cdot\hat{x})\,\hat{x} - \vec{p}}{4\pi\epsilon_0\, r^3}$$

NB: $\hat{x}$ is a unit vector along $\vec{x}$, not a Cartesian unit vector.

Note:

1) Multipole moments depend on choice of origin. For example, for a point charge at the origin, only $l = 0$ multipole is non-zero. If point charge is displaced from origin, higher order moments do not vanish.

2) The lowest non-vanishing multipole moment is independent of the origin.

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Consider a charge dist $\rho(\vec{x})$ localized to a volume *V* and subjected to a potential $\Phi(\vec{x})$ due to charges located outside of *V*. The potential energy of the charge dist in the external potential is

$$W = \int \rho(\vec{x})\,\Phi(\vec{x})\,d^3x$$

If $\Phi$ varies slowly over *V*, then Taylor expand it about an origin in *V*.

$$\begin{aligned}\Phi(\vec{x}) &= \Phi(0) + \vec{x}\cdot\nabla\Phi(0) + \frac{1}{2}\sum_{i,j} x_i\, x_j \,\frac{\partial^2\Phi}{\partial x_i\,\partial x_j}(0) + \ldots \\ &= \Phi(0) - \vec{x}\cdot\vec{E}(0) - \frac{1}{2}\sum_{i,j} x_i\, x_j \,\frac{\partial E_j}{\partial x_i}(0) + \ldots \\ &= \Phi(0) - \vec{x}\cdot\vec{E}(0) - \frac{1}{6}\sum_{i,j}(3x_i\, x_j - r^2\delta_{ij})\,\frac{\partial E_j}{\partial x_i}(0) + \ldots\end{aligned}$$

$$\left(\text{subtracting}\ \ \frac{1}{6}r^2\,\nabla\cdot\vec{E}(0) = \frac{1}{6}r^2\sum_i \frac{\partial E_i}{\partial x_i} = \frac{1}{6}r^2\sum_{i,j}\frac{\partial E_j}{\partial x_i}\delta_{ij}\ \ \text{from the last term}\right)$$

$\nabla\cdot\vec{E}(0) = 0$ since the charge that produces $\vec{E}(0)$ is located outside *V*.

Substituting this approx for *Φ* into the integral yields

$$W = q\,\Phi(0) - \vec{p}\cdot\vec{E}(0) - \frac{1}{6}\sum_{i,j} Q_{ij}\,\frac{\partial E_j}{\partial x_i}(0) + \ldots$$

As a simple example, the interaction energy between 2 dipoles is

$$W_{12} = -\vec{p}_1 \cdot \left[ \frac{3(\vec{p}_2 \cdot \hat{n})\,\hat{n} - \vec{p}_2}{4\pi\epsilon_0\,|\vec{x}_1 - \vec{x}_2|^3} \right]$$ where $\hat{n}$ points from 2 to 1, i.e., along $(\vec{x}_1 - \vec{x}_2)$

$$= \frac{\vec{p}_1 \cdot \vec{p}_2 - 3(\hat{n} \cdot \vec{p}_1)\,(\hat{n} \cdot \vec{p}_2)}{4\pi\epsilon_0\,|\vec{x}_1 - \vec{x}_2|^3}$$

SP 4.1, 4.2

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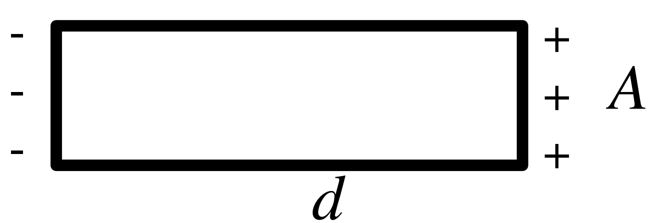
Jackson section 4.3:

When an electric field is applied to a dielectric, the dominant multipole moment of the response is the dipole.

$\vec{P}$ = dipole moment per unit volume

Consider a volume where $\vec{P}$ does not vary substantially; specifically a box with length $d$ (along the direction of the dipole moments) and end-face area $A$.




total dipole moment in volume $Ad$ :

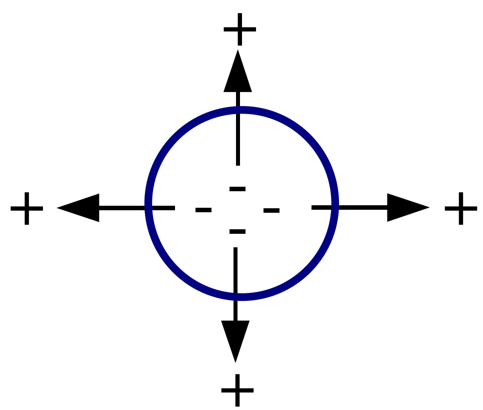
$$p = PAd = qd$$

$$\Rightarrow \quad \sigma_b = q/A = P$$

If end face is slanted at angle $\theta$, then area increases by 1/cos $\theta$

$$\Rightarrow \quad \sigma_b = P\cos\theta = \vec{P}\cdot\hat{n}$$

Diverging polarization yields bound charge pile-up:




Charge inside volume $= \int \rho_b dV$

= - charge driven past bounding surface $= -\int \vec{P}\cdot\hat{n}da = -\int(\nabla\cdot\vec{P})dV$

$$\Rightarrow \quad \rho_b = -\nabla\cdot\vec{P}$$

To see these results directly from the dipole potential:

potential due to polarization (slide 4):

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\vec{P}(\vec{x}')\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3} \quad = \frac{1}{4\pi\epsilon_0}\int d^3x' \vec{P}(\vec{x}')\cdot\nabla_{x'}\left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$$

$$= \frac{1}{4\pi\epsilon_0}\left\{\int d^3x' \nabla_{x'}\cdot\left[\frac{\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right] - \int d^3x' \frac{\nabla_{x'}\cdot\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right\}$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{\vec{P}(\vec{x}')\cdot\hat{n}da'}{|\vec{x}-\vec{x}'|} - \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\nabla_{x'}\cdot\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$$= \frac{1}{4\pi\epsilon_0}\int \frac{\sigma_b(\vec{x}')da'}{|\vec{x}-\vec{x}'|} + \frac{1}{4\pi\epsilon_0}\int d^3x' \frac{\rho_b(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} = \frac{\rho_f - \nabla \cdot \vec{P}}{\epsilon_0}$$

$\rho_f$ is the free-charge volume density

$$\nabla \cdot (\epsilon_0 \vec{E} + \vec{P}) = \rho_f$$

$$\nabla \cdot \vec{D} = \rho_f \quad \Rightarrow \quad \oint \vec{D} \cdot d\vec{a} = Q_{f,\,\mathrm{enc}}$$

with the electric displacement $\vec{D} \equiv \epsilon_0 \vec{E} + \vec{P}$

The equations of electrostatics in a dielectric are:

$$\nabla \times \vec{E} = 0 \qquad \nabla \cdot \vec{D} = \rho_f$$

For a linear, isotropic medium, $\vec{P} = \epsilon_0 \chi_e \vec{E}$

$\chi_e$ is the "electric susceptibility"

$$\Rightarrow \quad \vec{D} = \epsilon \vec{E} \quad \text{with} \quad \epsilon = \epsilon_0 (1 + \chi_e)$$

$\epsilon / \epsilon_0 = 1 + \chi_e$ is called the "dielectric constant".

For a linear dielectric: 

$$\rho_b = -\nabla \cdot \vec{P} = -\nabla \cdot [\vec{D} - \epsilon_0 \vec{E}] = -\nabla \cdot \left[\vec{D} - \frac{\epsilon_0}{\epsilon}\vec{D}\right]$$

$$= -\nabla \cdot \left[\frac{\epsilon - \epsilon_0}{\epsilon}\vec{D}\right] = -\frac{\epsilon - \epsilon_0}{\epsilon}\nabla \cdot \vec{D} = -\frac{\epsilon - \epsilon_0}{\epsilon}\rho_f$$

The final steps are for a uniform medium (i.e., $\epsilon$ doesn't vary with position). In this case,

$\nabla \cdot \vec{E} = \rho/\epsilon$ (The subscript "*f*" is often omitted.)

=> All problems in the medium are the same as in vacuum, except the field produced by a charge is reduced by factor $\epsilon_0/\epsilon$.

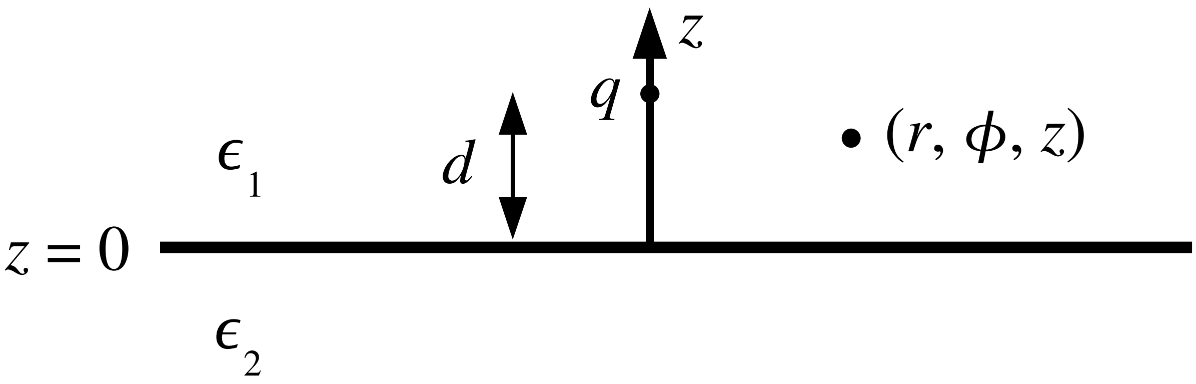
Boundary conditions at a surface with charge density $\sigma$ (not including polarization charge):

$$(\vec{D}_2 - \vec{D}_1) \cdot \hat{n}_{21} = \sigma$$

$$(\vec{E}_2 - \vec{E}_1) \times \hat{n}_{21} = 0$$

$\hat{n}_{21}$ points from medium 1 to medium 2.

Example (Jackson sec 4.4): Find $\Phi$ everywhere for




$$z > 0: \quad \nabla \cdot \vec{E} = \rho/\epsilon_1$$

$$z < 0: \quad \nabla \cdot \vec{E} = 0$$

$$\text{everywhere}: \quad \nabla \times \vec{E} = 0 \quad \Rightarrow \quad \vec{E} \text{ can be derived from a potential } \Phi$$

Boundary conditions: $D_\perp$ and $E_\parallel$ are continuous => at $z = 0$,

$$\epsilon_1 E_{1z} = \epsilon_2 E_{2z} \qquad (1)$$

$$E_{1,\parallel} = E_{2,\parallel} \qquad (2)$$

For $z > 0$, use the method of images: charge $q'$ at $z' = -d$

$$\Phi(z>0) = \frac{1}{4\pi\epsilon_1}\left[\frac{q}{\sqrt{r^2+(d-z)^2}} + \frac{q'}{\sqrt{r^2+(d+z)^2}}\right]$$

For $z < 0$, try a charge $q''$ at $z'' = d$

$$\Phi(z<0) = \frac{1}{4\pi\epsilon_2}\frac{q''}{\sqrt{r^2+(d-z)^2}}$$

$$\vec{E} = -\nabla\Phi \quad \Rightarrow \quad E_{1z}(z=0) = -\frac{1}{4\pi\epsilon_1}\frac{(q-q')d}{(r^2+d^2)^{3/2}}$$

$$E_{2z}(z=0) = -\frac{1}{4\pi\epsilon_2}\frac{q''d}{(r^2+d^2)^{3/2}}$$

$$(1) \quad \Rightarrow \quad q - q' = q'' \qquad (1b)$$

Tangential component is $E_r$

$$(2) \quad \Rightarrow \quad \epsilon_2(q+q') = \epsilon_1 q'' \qquad (2b)$$

(This also follows from continuity of $\Phi$ at $z = 0$.)

From (1b) and (2b), $$q' = \frac{\epsilon_1 - \epsilon_2}{\epsilon_1 + \epsilon_2} q \qquad q'' = \frac{2\epsilon_2}{\epsilon_1 + \epsilon_2} q$$

We've found a potential that satisfies Poisson's eqn everywhere and the boundary conditions.

Next, find the polarization charge on the plane $z = 0$:

$$\sigma_{\text{pol}} = \vec{P}_1 \cdot \hat{n}_1 + \vec{P}_2 \cdot \hat{n}_2 \quad = -\vec{P}_1 \cdot \hat{z} + \vec{P}_2 \cdot \hat{z}$$

$$= -(\epsilon_1 - \epsilon_0)E_{1,z} + (\epsilon_2 - \epsilon_0)E_{2,z}$$

$$= -\frac{q}{2\pi} \frac{\epsilon_0(\epsilon_2 - \epsilon_1)}{\epsilon_1(\epsilon_2 + \epsilon_1)} \frac{d}{(r^2 + d^2)^{3/2}}$$

## Electrostatic energy in dielectric media

Given charge density $\rho(\vec{x})$ in a region of space, the work done to bring in a differential amount of new charge, with density $\delta\rho(\vec{x})$, is

$$\delta W = \int \delta\rho(\vec{x})\,\Phi(\vec{x})\,d^3x$$

$\Phi$ is the potential due to the original charge $\rho$.

$$\nabla \cdot \vec{D} = \rho \quad \Rightarrow \quad \delta\rho = \nabla \cdot (\delta\vec{D})$$

$$\begin{aligned}
\Rightarrow \quad \delta W &= \int \Phi \, \nabla \cdot (\delta \vec{D}) \, d^3x \\
&= \int \left[ \nabla \cdot (\delta \vec{D} \, \Phi) - \delta \vec{D} \cdot \nabla \Phi \right] \, d^3x \\
&= \int \Phi \, \delta \vec{D} \cdot d\vec{a} + \int \vec{E} \cdot \delta \vec{D} \, d^3x
\end{aligned}$$

0 if $\rho$ is localized

$$\Rightarrow \quad W = \int d^3x \int_0^D \vec{E} \cdot \delta \vec{D} \qquad (1)$$

For a linear medium, $\vec{E} \cdot \delta \vec{D} = \epsilon \vec{E} \cdot \delta \vec{E} = \frac{1}{2} \epsilon \delta (\vec{E} \cdot \vec{E}) = \frac{1}{2} \delta (\vec{E} \cdot \vec{D})$

$$\begin{aligned}
\Rightarrow \quad W = \frac{1}{2} \int d^3x \int_0^D \delta (\vec{E} \cdot \vec{D}) &= \frac{1}{2} \int \vec{E} \cdot \vec{D} \, d^3x \\
&= -\frac{1}{2} \int \nabla \Phi \cdot \vec{D} \, d^3x \\
&= -\frac{1}{2} \int [\nabla \cdot (\Phi \vec{D}) - \Phi \, \nabla \cdot \vec{D}] \, d^3x \\
&= -\frac{1}{2} \int \Phi \, \vec{D} \cdot d\vec{a} + \frac{1}{2} \int \Phi \, \nabla \cdot \vec{D} \, d^3x = \frac{1}{2} \int \rho \, \Phi \, d^3x
\end{aligned}$$

If the dielectric is not linear, then it is not generally true that

$$W = \frac{1}{2}\int \rho\,\Phi\, d^3x$$

In this case, the detailed history of the system must be taken into account, using eq. (1).

SP 4.3—4.6