Magnetostatics; Faraday's Law; Quasi-Static Fields

Reading: Jackson 5.1 through 5.12, 5.15 through 5.18

Review of basic magnetostatics (i.e., cases with steady currents):

There are no magnetic monopoles.

Magnetic-flux density (or, magnetic induction) $\vec{B}$ is produced by currents.

Continuity eqn: $\frac{\partial \rho}{\partial t} + \nabla \cdot \vec{J} = 0$

In magnetostatics, $\nabla \cdot \vec{J} = 0$

$\vec{J}$ is current density: amount of positive charge crossing unit area per unit time

Biot-Savart Law: A current *I* flowing along a differential length element $d\vec{l}$ yields a differential flux density $d\vec{B}$ at position $\vec{x}$ (points from length element to observation point): $d\vec{B} = \frac{\mu_0}{4\pi} I \frac{d\vec{l} \times \vec{x}}{|\vec{x}|^3}$

$$I\, d\vec{l} = J\, da\, d\vec{l} = \vec{J}\, d^3x$$

Force on an element of current: $d\vec{F} = I\, d\vec{l} \times \vec{B} = \vec{J} \times \vec{B}\, d^3x$

Torque on a magnetic dipole: $\vec{N} = \vec{\mu} \times \vec{B}$

For a distribution of current with density $\vec{J}(\vec{x})$ exposed to an external $\vec{B}$,

$$\vec{F} = \int d\vec{F} = \int \vec{J} \times \vec{B}\, d^3x \qquad \vec{N} = \int \vec{x} \times (\vec{J} \times \vec{B})\, d^3x$$

Biot-Savart: $d\vec{B} = \dfrac{\mu_0}{4\pi} \dfrac{\vec{J} \times (\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|^3} d^3x'$ (field at $\vec{x}$ due to current at $\vec{x}'$)

$$\Rightarrow \quad \vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}') \times \frac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} d^3x'$$

Since $\dfrac{\vec{x} - \vec{x}'}{|\vec{x} - \vec{x}'|^3} = -\nabla_x \left(\dfrac{1}{|\vec{x} - \vec{x}'|} \right)$,

$$\vec{B}(\vec{x}) = -\frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}') \times \nabla_x \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) d^3x'$$

$$\vec{B}(\vec{x}) = -\frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}') \times \nabla_x \left(\frac{1}{|\vec{x}-\vec{x}'|}\right) d^3x'$$

Vector identity: $\nabla \times (\psi\vec{a}) = \nabla\psi \times \vec{a} + \psi\,\nabla \times \vec{a}$

$$\Rightarrow \quad \vec{a} \times \nabla\psi = \psi\,\nabla \times \vec{a} - \nabla \times (\psi\vec{a})$$

$$\Rightarrow \quad \vec{J}(\vec{x}') \times \nabla_x \left(\frac{1}{|\vec{x}-\vec{x}'|}\right) = \frac{1}{|\vec{x}-\vec{x}'|}\,\nabla_x \times \vec{J}(\vec{x}') - \nabla_x \times \left[\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right]$$

↑ $= 0$ since $\vec{J}(\vec{x}')$ doesn't depend on $\vec{x}$

$$\Rightarrow \quad \vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\int \nabla_x \times \left[\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right] d^3x' \quad = \frac{\mu_0}{4\pi}\nabla_x \times \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x'$$

Since $\nabla \cdot (\nabla \times \vec{a}) = 0$, $\quad \nabla \cdot \vec{B} = 0$

$$\nabla \times \vec{B} = \frac{\mu_0}{4\pi}\nabla \times \left[\nabla \times \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x'\right]$$

$$\nabla \times (\nabla \times \vec{a}) = \nabla(\nabla \cdot \vec{a}) - \nabla^2 \vec{a}$$

$$\nabla_x \times \vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\left[\nabla_x\left(\nabla_x \cdot \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x'\right) - \nabla_x^2 \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\, d^3x'\right]$$

$$= \frac{\mu_0}{4\pi}\nabla_x \int \nabla_x \cdot \left[\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right] d^3x' - \frac{\mu_0}{4\pi}\int \nabla_x^2 \left[\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right] d^3x'$$

Vector identity: $\nabla \cdot (\psi \vec{a}) = \vec{a} \cdot \nabla \psi + \psi\, \nabla \cdot \vec{a}$

$$\Rightarrow \quad \nabla_x \cdot \left[\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right] = \vec{J}(\vec{x}') \cdot \nabla_x \left(\frac{1}{|\vec{x}-\vec{x}'|}\right) + \frac{1}{|\vec{x}-\vec{x}'|}\nabla_x \cdot \vec{J}(\vec{x}')$$

$= 0$ since $\vec{J}(\vec{x}')$ doesn't depend on $\vec{x}$

Also, $\nabla_x^2 \left[\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right] = \vec{J}(\vec{x}')\, \nabla_x^2 \left(\frac{1}{|\vec{x}-\vec{x}'|}\right)$ =>

$$\nabla_x \times \vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\nabla_x \int \vec{J}(\vec{x}') \cdot \nabla_x \left(\frac{1}{|\vec{x}-\vec{x}'|}\right) d^3x' - \frac{\mu_0}{4\pi}\int \vec{J}(\vec{x}')\, \nabla_x^2 \left(\frac{1}{|\vec{x}-\vec{x}'|}\right) d^3x'$$

$$\nabla_x \times \vec{B}(\vec{x}) = \frac{\mu_0}{4\pi} \nabla_x \int \vec{J}(\vec{x}') \cdot \nabla_x \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) d^3x' - \frac{\mu_0}{4\pi} \int \vec{J}(\vec{x}')\, \nabla_x^2 \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) d^3x'$$

$$\nabla_x \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) = -\nabla_{x'} \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) \quad \text{and} \quad \nabla_x^2 \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) = -4\pi\delta(\vec{x} - \vec{x}')$$

$$\Rightarrow \quad \nabla_x \times \vec{B}(\vec{x}) = -\frac{\mu_0}{4\pi} \nabla_x \int \vec{J}(\vec{x}') \cdot \nabla_{x'} \left(\frac{1}{|\vec{x} - \vec{x}'|} \right) d^3x' + \mu_0 \int \vec{J}(\vec{x}')\, \delta(\vec{x} - \vec{x}')\, d^3x'$$

$$= -\frac{\mu_0}{4\pi} \nabla_x \int \left\{ \nabla_{x'} \cdot \left[\frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \right] - \frac{\nabla_{x'} \cdot \vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \right\} d^3x' + \mu_0 \vec{J}(\vec{x})$$

$\nabla \cdot \vec{J} = 0$ in magnetostatics

$$\Rightarrow \quad \nabla_x \times \vec{B}(\vec{x}) = -\frac{\mu_0}{4\pi} \nabla_x \int \frac{\vec{J}(\vec{x}') \cdot d\vec{a}'}{|\vec{x} - \vec{x}'|} + \mu_0 \vec{J}(\vec{x}) \quad \Rightarrow \quad \nabla \times \vec{B} = \mu_0 \vec{J}$$

↑ = 0 for a localized current dist

Consider an open surface S bounded by a closed curve C.

$$\int_S (\nabla \times \vec{B}) \cdot d\vec{a} = \mu_0 \int_S \vec{J} \cdot d\vec{a}$$

Stokes's Thm => $\oint_C \vec{B} \cdot d\vec{l} = \mu_0 I$ (Ampère's Law)

The Vector Potential (Jackson sec 5.4)

We want to solve the eqns of magnetostatics:

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{B} = \mu_0 \vec{J}$$

If $\vec{J} = 0$ in the region of interest, then we can introduce a magnetic scalar potential Φ_M and $\vec{B} = -\nabla\Phi_M$. In this case, we can use the techniques of electrostatics for solving Laplace's eqn (more later).

More generally, $\nabla \cdot \vec{B} = 0 \quad \Rightarrow \quad \vec{B}(\vec{x}) = \nabla \times \vec{A}(\vec{x})$

$\vec{A}(\vec{x})$ is the "vector potential".

From Jackson 5.16 (or, p 3 of lecture notes),

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' + \nabla\Psi(\vec{x})$$

[2nd term is OK since $\nabla \times (\nabla\Psi) = 0$.]

$\vec{A} \to \vec{A} + \nabla\Psi$ is called a "gauge transformation".

We can use gauge transformations to yield a convenient form for $\nabla \cdot \vec{A}$.

$$\nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J} \quad \Rightarrow \quad \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = \mu_0 \vec{J}$$

Coulomb gauge: $\nabla \cdot \vec{A} = 0$

$\Rightarrow \quad \nabla^2 \vec{A} = -\mu_0 \vec{J}$ (each Cartesian component of $\vec{A}$ satisfies Poisson's eqn)

$$\nabla_x \cdot \vec{A}(\vec{x}) = \nabla_x \cdot \left[\frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}d^3x' + \nabla_x\Psi(\vec{x})\right]$$

$$= \frac{\mu_0}{4\pi}\int \nabla_x \cdot \left[\frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}\right] d^3x' + \nabla_x^2\Psi(\vec{x})$$

$$\int \nabla_x \cdot \left[\frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \right] d^3x' = 0$$

for all space if the current dist is localized (from slides 4 and 5)

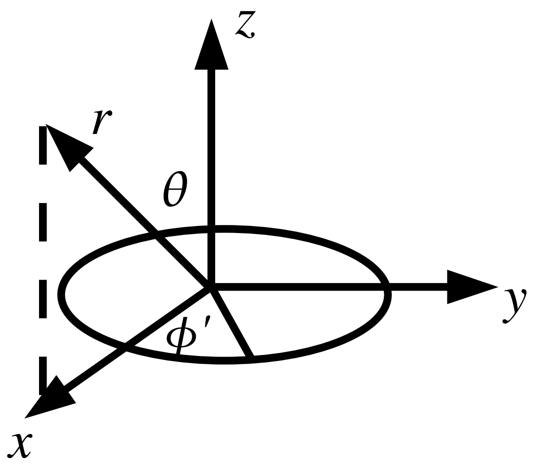
So, $\nabla \cdot \vec{A} = 0 \quad \Rightarrow \quad \nabla^2 \Psi = 0$

=> Ψ = const   for Coulomb gauge applied to all space

$$\Rightarrow \quad \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'$$

Analogous to scalar potential for all space: $\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x'$

A Circular Current Loop (Jackson sec 5.5)



The loop has radius a, lies in the x-y plane, is centered on the origin, and carries current I.

Spherical coords: $\vec{J}$ is in the $\hat{\phi}$ direction.

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}\, d^3x'$$

$$\vec{J}(\vec{x}')\, d^3x' = I\, d\vec{l}' = I\, a\, d\phi' \,(-\sin\phi'\,\hat{x} + \cos\phi'\,\hat{y})$$

Cylindrical symmetry => freedom to place observation point in the *x*-*z* plane ($\phi = 0$)

$$\begin{aligned}
|\vec{x} - \vec{x}'|^2 &= (x - x')^2 + (y - y')^2 + (z - z')^2 \\
&= (r \sin\theta - a\cos\phi')^2 + (0 - a\sin\phi')^2 + (r\cos\theta - 0)^2 \\
&= r^2 \sin^2\theta + a^2\cos^2\phi' - 2ar\sin\theta\cos\phi' + a^2\sin^2\phi' + r^2\cos^2\theta \\
&= r^2 + a^2 - 2ar\sin\theta\cos\phi'
\end{aligned}$$

$$\vec{A}(\vec{x}) = \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \frac{d\phi' \, (-\sin\phi' \, \hat{x} + \cos\phi' \, \hat{y})}{\sqrt{r^2 + a^2 - 2ar\sin\theta\cos\phi'}}$$

The $\hat{x}$ integral vanishes

(substitute $u = r^2 + a^2 - 2ar\sin\theta\cos\phi'$ and the endpoints are the same)

$\Rightarrow \quad \vec{A}(\vec{x})$ is in the $\hat{y}$ direction, which is also the $\hat{\phi}$ direction.

So, $A_r = 0$, $A_\theta = 0$, and $\quad A_\phi(r,\theta) = \frac{\mu_0 I a}{4\pi} \int_0^{2\pi} \frac{\cos\phi' \, d\phi'}{\sqrt{r^2 + a^2 - 2ar\sin\theta\cos\phi'}}$

If $r \gg a$, then $(r^2 + a^2 - 2ar\sin\theta\cos\phi')^{-1/2} \approx \frac{1}{r}\left(1 + \frac{a}{r}\sin\theta\cos\phi'\right)$

$$\Rightarrow \quad A_\phi = \frac{\mu_0 I a}{4\pi r} \int_0^{2\pi} d\phi' \, \cos\phi' \left(1 + \frac{a}{r}\sin\theta\cos\phi'\right) \quad = \frac{\mu_0 I a^2 \sin\theta}{4\pi r^2} \, \pi$$

$$\vec{B} = \nabla \times \vec{A}$$

$$B_r = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, A_\phi) = \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{\mu_0\pi a^2 I}{4\pi r^2}\sin^2\theta\right) = \frac{\mu_0}{2\pi}(\pi a^2 I)\frac{\cos\theta}{r^3}$$

$$B_\theta = -\frac{1}{r}\frac{\partial}{\partial r}(rA_\phi) = -\frac{1}{r}\frac{\partial}{\partial r}\left(\frac{\mu_0\pi a^2 I\sin\theta}{4\pi}r^{-1}\right) = \frac{\mu_0}{4\pi}(\pi a^2 I)\frac{\sin\theta}{r^3}$$

$$B_\phi = 0$$

Recall electric dipole fields:

$$E_r = \frac{1}{2\pi\epsilon_0}p\frac{\cos\theta}{r^3} \qquad E_\theta = \frac{1}{4\pi\epsilon_0}p\frac{\sin\theta}{r^3} \qquad E_\phi = 0$$

=> magnetic field has dipole character,

with magnetic dipole moment $m = \pi a^2 I$

Next, consider the magnetic induction far from a localized, arbitrary current dist (Jackson sec 5.6).

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi}\int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'$$

$$\begin{aligned}
\frac{1}{|\vec{x}-\vec{x}'|} &= \left(|\vec{x}|^2 + |\vec{x}'|^2 - 2\vec{x}\cdot\vec{x}'\right)^{-1/2} \\
&= \frac{1}{|\vec{x}|}\left(1 + \frac{|\vec{x}'|^2}{|\vec{x}|^2} - \frac{2\vec{x}\cdot\vec{x}'}{|\vec{x}|^2}\right)^{-1/2} \\
&= \frac{1}{|\vec{x}|}\left(1 + \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^2} + ...\right) \\
&= \frac{1}{|\vec{x}|} + \frac{\vec{x}\cdot\vec{x}'}{|\vec{x}|^3} + ...
\end{aligned}$$

So, each Cartesian component is

$$A_i(\vec{x}) = \frac{\mu_0}{4\pi}\left[\frac{1}{|\vec{x}|}\int J_i(\vec{x}')\, d^3x' + \frac{\vec{x}}{|\vec{x}|^3}\cdot\int J_i(\vec{x}')\,\vec{x}'\, d^3x' + ...\right]$$

To further simplify, we first derive a useful result. Suppose $f(\vec{x}')$ and $g(\vec{x}')$ are arbitrary functions and $\vec{J}(\vec{x}')$ is localized (but not necessarily divergenceless).

$$\int f g\, \vec{J} \cdot d\vec{a}' = 0 \qquad \text{for a localized current dist}$$

$$\Rightarrow \quad \int \nabla_{x'} \cdot (f g\, \vec{J})\, d^3x' = 0$$

$$\nabla_{x'} \cdot (f g\, \vec{J}) = \vec{J} \cdot \nabla_{x'}(f g) + f g \nabla_{x'} \cdot \vec{J}$$

$$= f\, \vec{J} \cdot \nabla_{x'} g + g\, \vec{J} \cdot \nabla_{x'} f + f g \nabla_{x'} \cdot \vec{J}$$

=>

$$\int \left(f\, \vec{J} \cdot \nabla_{x'} g + g\, \vec{J} \cdot \nabla_{x'} f + f g \nabla_{x'} \cdot \vec{J} \right) d^3x' = 0 \qquad (1)$$

Now, take $f = 1$, $g = x_i'$, and impose $\nabla_{x'} \cdot \vec{J} = 0$

$$\Rightarrow \quad \int \left(\vec{J} \cdot \nabla_{x'} x'_i + x'_i \, \vec{J} \cdot \nabla_{x'}(1) + x'_i \, \nabla_{x'} \cdot \vec{J} \right) d^3x' = 0$$

$$= 0 \qquad\qquad = 0$$

$$\nabla_{x'} x'_i = \frac{\partial x'_i}{\partial x'_1} \, \hat{x}'_1 + \frac{\partial x'_i}{\partial x'_2} \, \hat{x}'_2 + \frac{\partial x'_i}{\partial x'_3} \, \hat{x}'_3 = \hat{x}'_i$$

$$\Rightarrow \quad \int \vec{J} \cdot \hat{x}'_i \, d^3x' = \int J_i(\vec{x}') \, d^3x' = 0$$

$$\Rightarrow \quad A_i(\vec{x}) \approx \frac{\mu_0}{4\pi} \, \frac{\vec{x}}{|\vec{x}|^3} \cdot \int J_i(\vec{x}') \, \vec{x}' \, d^3x'$$

With $f = x'_i$, $g = x'_j$, and $\nabla_{x'} \cdot \vec{J} = 0$ in (1),

$$\int \left(x'_i \, \vec{J} \cdot \nabla_{x'} x'_j + x'_j \, \vec{J} \cdot \nabla_{x'} x'_i \right) d^3x' = 0$$

$$\Rightarrow \quad \int \left(x'_i \, J_j + x'_j \, J_i \right) d^3x' = 0$$

$$\vec{x} \cdot \int \vec{x}'\, J_i\, d^3x' = \sum_j x_j \int x_j'\, J_i\, d^3x'$$

$$= \sum_j x_j \int \left[x_j'\, J_i - \frac{1}{2} \left(x_i'\, J_j + x_j'\, J_i \right) \right] d^3x'$$

$$= -\frac{1}{2} \sum_j x_j \int \left(x_i'\, J_j - x_j'\, J_i \right) d^3x'$$

$$i = 1 \;\; : \quad \sum_j x_j \left(x_i'\, J_j - x_j'\, J_i \right)$$

$$= x_1 \left(x_1'\, J_1 - x_1'\, J_1 \right) + x_2 \left(x_1'\, J_2 - x_2'\, J_1 \right) + x_3 \left(x_1'\, J_3 - x_3'\, J_1 \right)$$

$$= [\vec{x} \times (\vec{x}' \times \vec{J})]_1$$

Likewise for $i = 2, 3$

$$\Rightarrow \quad \vec{x} \cdot \int J_i(\vec{x}')\, \vec{x}'\, d^3x' = -\frac{1}{2} \left[\vec{x} \times \int \vec{x}' \times \vec{J}(\vec{x}')\, d^3x' \right]_i \qquad (2)$$

$$\Rightarrow \quad \vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \left(-\frac{1}{2} \right) \frac{1}{|\vec{x}|^3}\, \vec{x} \times \int \vec{x}' \times \vec{J}(\vec{x}')\, d^3x'$$

Define the magnetic moment density, or magnetization, as

$$\vec{M}(\vec{x}) = \frac{1}{2}\,\vec{x} \times \vec{J}(\vec{x})$$

magnetic moment $\quad \vec{m} = \int \vec{M}(\vec{x}')\, d^3x' = \frac{1}{2} \int \vec{x}' \times \vec{J}(\vec{x}')\, d^3x'$

$$\Rightarrow \quad \vec{A}(\vec{x}) \approx \frac{\mu_0}{4\pi}\,\frac{(-1)}{|\vec{x}|^3}\,\vec{x} \times \vec{m} = \frac{\mu_0}{4\pi}\,\frac{\vec{m} \times \vec{x}}{|\vec{x}|^3}$$

$$\vec{B} = \nabla \times \vec{A} = \frac{\mu_0}{4\pi}\,\nabla \times \left(\frac{\vec{m} \times \vec{x}}{|\vec{x}|^3}\right)$$

Using lots of identities on the cover of Jackson, and with
$\hat{n}$ = a unit vector along $\vec{x}$:

$$\nabla \times \left(\frac{\vec{m} \times \vec{x}}{|\vec{x}|^3}\right) = \nabla \left(\frac{1}{|\vec{x}|^3}\right) \times (\vec{m} \times \vec{x}) + \frac{1}{|\vec{x}|^3}\,\nabla \times (\vec{m} \times \vec{x})$$

$$= -\frac{3}{|\vec{x}|^4}\,\hat{n} \times (\vec{m} \times \vec{x}) + \frac{1}{|\vec{x}|^3}\left[\vec{m}\,\nabla \cdot \vec{x} - \vec{x}\,\nabla \cdot \vec{m} + (\vec{x} \cdot \nabla)\,\vec{m} - (\vec{m} \cdot \nabla)\,\vec{x}\right]$$

$$= -\frac{3}{|\vec{x}|^4}\,\hat{n}\times(\vec{m}\times\vec{x}) + \frac{1}{|\vec{x}|^3}\left[\vec{m}\,\underbrace{\nabla\cdot\vec{x}}_{=3} - \vec{x}\,\underbrace{\nabla\cdot\vec{m}}_{=0} + \underbrace{(\vec{x}\cdot\nabla)\,\vec{m}}_{=0} - (\vec{m}\cdot\nabla)\,\vec{x}\right]$$

$$= -\frac{3}{|\vec{x}|^4}\left[(\hat{n}\cdot\vec{x})\,\vec{m} - (\hat{n}\cdot\vec{m})\,\vec{x}\right] + \frac{1}{|\vec{x}|^3}\left[3\vec{m} - \vec{m}\right]$$

$$= \frac{1}{|\vec{x}|^3}\left[-3\vec{m} + 3\hat{n}\,(\hat{n}\cdot\vec{m}) + 3\vec{m} - \vec{m}\right] = \frac{1}{|\vec{x}|^3}\left[3\hat{n}\,(\hat{n}\cdot\vec{m}) - \vec{m}\right]$$

$$\Rightarrow \quad \vec{B}(\vec{x}) = \frac{\mu_0}{4\pi}\,\frac{3\hat{n}\,(\hat{n}\cdot\vec{m}) - \vec{m}}{|\vec{x}|^3}$$

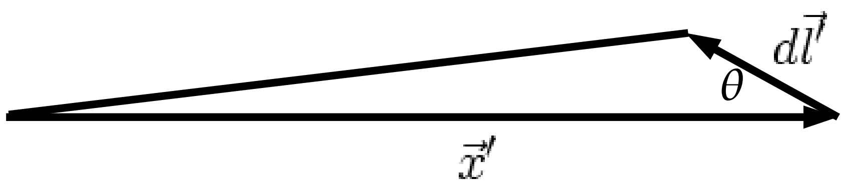
This has the same form as the field due to an electric dipole:

$$\vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0}\,\frac{3\hat{n}\,(\hat{n}\cdot\vec{p}) - \vec{p}}{|\vec{x}|^3}$$

So, any localized current dist produces a dipole magnetic induction to first order at large distances.

If the current is restricted to a single plane loop (of arbitrary shape),
then $\vec{J}(\vec{x}')\, d^3x' = I\, d\vec{l}'$

$$\Rightarrow \quad \vec{m} = \frac{1}{2}\int \vec{x}' \times \vec{J}(\vec{x}')\, d^3x' = \frac{I}{2}\oint \vec{x}' \times d\vec{l}'$$



$dl' \sin\theta =$ height of triangle

$x' =$ base length

$$\frac{1}{2}|\vec{x}' \times d\vec{l}'| = \frac{1}{2} x'\, dl' \sin\theta \quad = \text{area of triangle}$$

$\Rightarrow \quad |\vec{m}| = I \cdot \text{area} \;;\; \vec{m}$ points $\perp$ to the plane (use right hand rule with current)

Note that the circular current loop discussed earlier is a specific case of this general result (see slide 12).

Force and torque on (and energy of) a localized current distribution in an external magnetic induction $\vec{B}(\vec{x})$ (Jackson sec 5.7)

Brief mathematical prelude on the Levi-Civita tensor ϵ_{ijk}

ϵ_{ijk} = 1 for i, j, k = 1, 2, 3
2, 3, 1
3, 1, 2

-1 for i, j, k = 1, 3, 2
2, 1, 3
3, 2, 1

0 otherwise (i.e., for 2 or more indices equal)

$$(\vec{a} \times \vec{b})_i = \sum_{jk} \epsilon_{ijk} a_j b_k$$

For example, take i=1:

$$\sum_{jk} \epsilon_{1jk} a_j b_k = \epsilon_{123} a_2 b_3 + \epsilon_{132} a_3 b_2$$

$$= a_2 b_3 - a_3 b_2 = (\vec{a} \times \vec{b})_1$$

$$\vec{F} = \int \vec{J}(\vec{x}) \times \vec{B}(\vec{x}) d^3x$$

k^{th} component of the induction: $B_k(\vec{x}) = B_k(0) + \vec{x} \cdot \nabla B_k(0) + \ldots$

i^{th} component of force:

$$F_i = \int [\vec{J} \times \vec{B}]_i d^3x \quad = \int \sum_{jk} \epsilon_{ijk} J_j(\vec{x}) B_k(\vec{x}) d^3x$$

$$= \sum_{jk} \epsilon_{ijk} \int J_j(\vec{x}) \; [B_k(0) + \vec{x} \cdot \nabla B_k(0) + \ldots] \; d^3x$$

$$= \sum_{jk} \epsilon_{ijk} \left[B_k(0) \int J_j(\vec{x}) \, d^3x + \nabla B_k(0) \cdot \int J_j(\vec{x}) \, \vec{x} \, d^3x \right]$$

↑ = 0 for steady, localized current dist

(see slide 15)

Eq (2) on slide 16: $\vec{x} \cdot \int J_i(\vec{x}') \, \vec{x}' \, d^3x' = -\frac{1}{2} \left[\vec{x} \times \int \vec{x}' \times \vec{J}(\vec{x}') \, d^3x' \right]_i$

Replace $\vec{x} \to \nabla B_k(0)$, $\vec{x}' \to \vec{x}$, and $i \to j$

$$\Rightarrow \quad \nabla B_k(0) \cdot \int J_j(\vec{x})\, \vec{x}\, d^3x \;=\; -\frac{1}{2}\left[\nabla B_k(0) \times \int \vec{x} \times \vec{J}(\vec{x})\, d^3x\right]_j$$

$$= -\left[\nabla B_k(0) \times \vec{m}\right]_j$$

$$= \left[\vec{m} \times \nabla B_k(0)\right]_j$$

$$\Rightarrow \quad F_i = \sum_{jk} \epsilon_{ijk} \left[\vec{m} \times \nabla B_k(0)\right]_j \quad = \sum_{jk} \epsilon_{ijk} \left(\vec{m} \times \nabla\right)_j B_k(\vec{x})\Big]_{\vec{x}=0}$$

For example, $(\vec{m} \times \nabla)_1 B_k(\vec{x})]_{\vec{x}=0} = \left(m_2 \frac{\partial}{\partial x_3} - m_3 \frac{\partial}{\partial x_2}\right) B_k(\vec{x})]_{\vec{x}=0}$

$$= m_2 \frac{\partial B_k}{\partial x_3}(0) - m_3 \frac{\partial B_k}{\partial x_2}(0)$$

$$= \left[\vec{m} \times \nabla B_k(0)\right]_1$$

$$\Rightarrow \quad \vec{F} = (\vec{m} \times \nabla) \times \vec{B}\Big]_{\vec{x}=0} \quad = \left[\nabla(\vec{m} \cdot \vec{B}) - \vec{m}\, \nabla \cdot \vec{B}\right]_{\vec{x}=0}$$

(from Jackson cover; explicitly verified on next page)

=> $\quad \vec{F} = \nabla(\vec{m} \cdot \vec{B})\Big]_{\vec{x}=0}$

Explicit verification, for *i*=1:

$$
\begin{aligned}
F_1 &= \sum_{jk} \epsilon_{1jk}\,(\vec{m}\times\nabla)_j\,B_k \\
&= (\vec{m}\times\nabla)_2\,B_3 - (\vec{m}\times\nabla)_3\,B_2 \\
&= m_3\,\frac{\partial B_3}{\partial x_1} - m_1\,\frac{\partial B_3}{\partial x_3} - m_1\,\frac{\partial B_2}{\partial x_2} + m_2\,\frac{\partial B_2}{\partial x_1} \\
&= m_1\left(\frac{\partial B_1}{\partial x_1} - \nabla\cdot\vec{B}\right) + m_2\,\frac{\partial B_2}{\partial x_1} + m_3\,\frac{\partial B_3}{\partial x_1} \\
&= \frac{\partial}{\partial x_1}(m_1\,B_1 + m_2\,B_2 + m_3\,B_3) - m_1\,(\nabla\cdot\vec{B}) \\
&= \left[\nabla(\vec{m}\cdot\vec{B}) - \vec{m}\,(\nabla\cdot\vec{B})\right]_1
\end{aligned}
$$

$$\vec{F} = \nabla(\vec{m}\cdot\vec{B})\Big]_{\vec{x}=0}$$

So, to get the lowest-order force on a current dist due to an external $\vec{B}$:

1) Pick an origin within the dist, 2) Compute $\vec{m}$ wrt that origin,

3) Take the derivative $\nabla(\vec{m}\cdot\vec{B})$, 4) Evaluate at the origin.

Torque:
$$\vec{N} = \int \vec{x} \times (\vec{J} \times \vec{B})\, d^3x$$
$$\approx \int \vec{x} \times [\vec{J} \times \vec{B}(0)]\, d^3x$$
$$= \int [\vec{x} \cdot \vec{B}(0)]\, \vec{J}\, d^3x - \vec{B}(0) \int \vec{x} \cdot \vec{J}\, d^3x$$

Since $\vec{x} = r\hat{r}$, the second integral is $\int r\, J_r\, d^3x$

Recall eq 1 on slide 14: $\int \left(f\, \vec{J} \cdot \nabla_{x'} g + g\, \vec{J} \cdot \nabla_{x'} f + f g \nabla_{x'} \cdot \vec{J} \right) d^3x' = 0$

Take $f = g = r$, $\nabla \cdot \vec{J} = 0$; $\nabla r = \hat{r}$

$$\Rightarrow \quad \int (r\, J_r + r\, J_r)\, d^3x = 0 \quad \Rightarrow \quad \int \vec{x} \cdot \vec{J}\, d^3x = 0$$

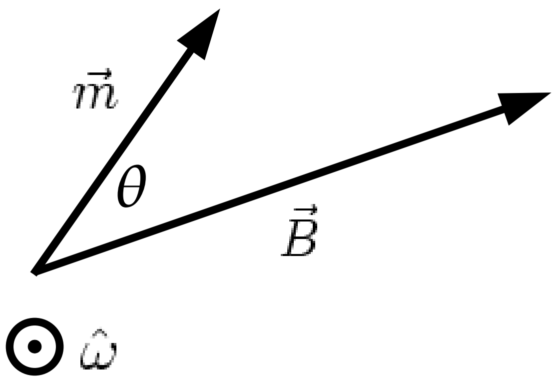
On top of slide 22, we found $\int [\vec{x} \cdot \nabla B_k(0)]\, \vec{J}\, d^3x = \vec{m} \times \nabla B_k(0)$

Replace $\nabla B_k(0)$ with $\vec{B}(0)$ $\Rightarrow$ $\int [\vec{x} \cdot \vec{B}(0)]\, \vec{J}\, d^3x = \vec{m} \times \vec{B}(0)$

$\Rightarrow \quad \vec{N} = \vec{m} \times \vec{B}(0)$ to lowest order

Potential energy of a permanent magnetic dipole in an external induction:

$$\vec{F} = -\nabla U = \nabla(\vec{m} \cdot \vec{B}) \qquad \Rightarrow \qquad U = -\vec{m} \cdot \vec{B}$$



Also, $\vec{N} = -\frac{\partial U}{\partial \theta}\,\hat{\omega} = \frac{\partial}{\partial \theta}(mB\cos\theta)\,\hat{\omega} = -mB\sin\theta\,\hat{\omega} = \vec{m} \times \vec{B}$

$\Rightarrow \quad \vec{m}$ tends to align along $\vec{B}$ (compass)

SP 5.1—5.5

Macroscopic Equations and Boundary Conditions (Jackson sec 5.8)

Magnetic dipole moment of matter comes from 1) current of electrons and 2) intrinsic magnetic moments of atoms.

As with electrostatics, averaging of the microscopic eqns yields the macroscopic eqns:

$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{H} = \vec{J}$ (only free current is included here)

$\vec{H}$ is called the "magnetic field".

$\vec{H} \equiv \frac{1}{\mu_0}\vec{B} - \vec{M}$ ($\vec{M}$ is the magnetization, i.e., dipole moment per unit volume)

effective current density $\vec{J}_M = \nabla \times \vec{M}$

(Jackson p. 192; analogous to electrostatic development in Topic 4, slides 8 through 10)

For linear, isotropic, paramagnetic and diamagnetic materials,

$\vec{B} = \mu \vec{H}$ (μ is called the "magnetic permeability")

More complicated for ferromagnetic materials, where $\vec{M}$ depends on history and may be non-zero even for zero applied induction.

Boundary conditions:

$$(\vec{B}_2 - \vec{B}_1) \cdot \hat{n} = 0 \qquad \hat{n} \times (\vec{H}_2 - \vec{H}_1) = \vec{K}$$

$\vec{K}$ = free surface current density (i.e., current per unit transverse length)

First eqn yields $\vec{H}_2 \cdot \hat{n} = \frac{\mu_1}{\mu_2} \vec{H}_1 \cdot \hat{n}$

If $\mu_1 \gg \mu_2$, then $\vec{H}_2$ has a much larger normal than tangential component (as long as $\vec{K}$ is not huge) => $\vec{H}_2$ is normal to the boundary surface, just as an external electric field is at a conducting surface.

Magnetic Boundary-Value Problems when $\vec{J} = 0$ (Jackson sec 5.9)

$$\nabla \times \vec{H} = 0 \quad \Rightarrow \quad \vec{H} = -\nabla\Phi_M$$ (magnetic scalar potential)

$\nabla \cdot \vec{B} = 0$ For linear media => $\nabla \cdot (\mu\vec{H}) = 0$

$$\Rightarrow \quad \nabla \cdot (\mu\nabla\Phi_M) = 0$$

If μ is piecewise constant, then

$\nabla^2\Phi_M = 0$ in each region of constant μ.

For "hard ferromagnets", $\vec{M}$ is fixed (doesn't depend on applied field).

$$\nabla \cdot \vec{B} = \nabla \cdot [\mu_0(\vec{H} + \vec{M})] = \mu_0 \nabla \cdot [-\nabla\Phi_M + \vec{M}] = 0$$

$$\Rightarrow \quad \nabla^2\Phi_M = -\rho_M$$

where the effective magnetic-charge density $\rho_M = -\nabla \cdot \vec{M}$

If there are no boundary surfaces, then in analogy with electrostatics

$$\Phi_M(\vec{x}) = -\frac{1}{4\pi} \int \frac{\nabla_{x'} \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|}\, d^3x'$$

If $\vec{M}$ is well-behaved and localized, then integration by parts yields

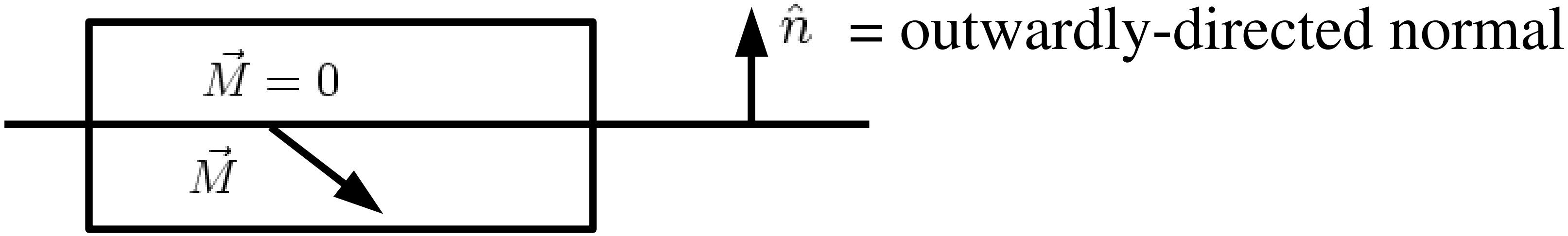
$$\Phi_M(\vec{x}) = -\frac{1}{4\pi} \nabla \cdot \int \frac{\vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|}\, d^3x'$$

Far from the region of non-vanishing magnetization, $\Phi_M(\vec{x}) \approx \frac{\vec{m} \cdot \vec{x}}{4\pi r^3}$

$\vec{m} = \int \vec{M}(\vec{x}')\, d^3x'$ = magnetic dipole moment

Suppose $\vec{M}$ is discontinuous: $\vec{M} \neq 0$ inside the ferromagnet and $\vec{M} = 0$ outside.

Gaussian pillbox straddling the surface:



"Magnetic charge" inside $q_M = -\int \nabla \cdot \vec{M}\, d^3x = -\int \vec{M} \cdot d\vec{a} = (-\vec{M}) \cdot (-\hat{n})\, A$

=> effective magnetic surface-charge density $\sigma_M = \vec{M} \cdot \hat{n}$

$$\Rightarrow \quad \Phi_M(\vec{x}) = -\frac{1}{4\pi} \int_V \frac{\nabla_{x'} \cdot \vec{M}(\vec{x}')}{|\vec{x} - \vec{x}'|} d^3x' + \frac{1}{4\pi} \oint_S \frac{\vec{M}(\vec{x}') \cdot \hat{n}'\, da'}{|\vec{x} - \vec{x}'|}$$

SP 5.6

Faraday's Law (Jackson sec 5.15)

If a closed curve C bounds surface S, then $\oint_C \vec{E}' \cdot d\vec{l} = -\frac{d}{dt}\int_S \vec{B} \cdot \hat{n}\, da$

where $\vec{E}'$ is the electric field at $d\vec{l}$ in its rest frame. The time derivative is a total derivative, accounting both for time variations in $\vec{B}$ and motion of the loop in space.

Adopting the rest frame of the loop:

$$\oint_C \vec{E}' \cdot d\vec{l} = \oint_C \vec{E} \cdot d\vec{l} = \int_S (\nabla \times \vec{E}) \cdot \hat{n}\, da$$

$$\Rightarrow \quad \int_S \left[\left(\nabla \times \vec{E}\right) + \frac{\partial \vec{B}}{\partial t} \right] \cdot \hat{n}\, da = 0 \qquad \Rightarrow \quad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

This generalizes $\nabla \times \vec{E} = 0$ for static fields to the dynamic case.

$\oint_C \vec{E} \cdot d\vec{l}$ is called the "electromotive force" (emf)

Magnetic Field Energy (Jackson sec 5.16)

The work done to generate the currents that yield a static magnetic field is

$$W = \int d^3x \int_0^B \vec{H} \cdot \delta\vec{B}$$

For a linear medium, $W = \frac{1}{2} \int \vec{H} \cdot \vec{B} \, d^3x$

For a localized current distribution,

$$\begin{aligned} W &= \frac{1}{2} \int \vec{H} \cdot (\nabla \times \vec{A}) \, d^3x \\ &= \frac{1}{2} \int [\vec{A} \cdot (\nabla \times \vec{H}) - \nabla \cdot (\vec{H} \times \vec{A})] \, d^3x \\ &= \frac{1}{2} \int \vec{J} \cdot \vec{A} \, d^3x \end{aligned}$$

Inductance (Jackson sec 5.17)

Consider N current-carrying circuits (the i^{th} one has current I_i) in vacuum.

$$\vec{J}(\vec{x}) = \sum_{i=1}^{N} \vec{J}_i(\vec{x})$$

$$W = \frac{1}{2}\int \vec{J}\cdot\vec{A}\,d^3x = \frac{\mu_0}{8\pi}\int d^3x \int d^3x' \,\frac{\vec{J}(\vec{x})\cdot\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$$= \frac{\mu_0}{8\pi}\int d^3x \int d^3x' \left[\sum_{i=1}^{N}\vec{J}_i(\vec{x})\right]\cdot\left[\sum_{j=1}^{N}\vec{J}_j(\vec{x}')\right]\frac{1}{|\vec{x}-\vec{x}'|}$$

$$= \frac{\mu_0}{8\pi}\sum_{i=1}^{N}\int d^3x \sum_{j=1}^{N}\int d^3x' \,\frac{\vec{J}_i(\vec{x})\cdot\vec{J}_j(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$$= \frac{1}{2}\sum_{i=1}^{N} L_i\, I_i^2 + \sum_{i=1}^{N}\sum_{j\neq i} M_{ij}\, I_i\, I_j$$

with

$$L_i = \frac{\mu_0}{4\pi I_i^2} \int_{C_i} d^3x \int_{C_i} d^3x' \, \frac{\vec{J}_i(\vec{x}) \cdot \vec{J}_i(\vec{x}')}{|\vec{x} - \vec{x}'|}$$ ("self-inductance")

$$M_{ij} = \frac{\mu_0}{4\pi I_i I_j} \int_{C_i} d^3x \int_{C_j} d^3x' \, \frac{\vec{J}_i(\vec{x}) \cdot \vec{J}_j(\vec{x}')}{|\vec{x} - \vec{x}'|}$$ ("mutual inductance")

Note : $M_{ji} = M_{ij}$

If the circuits are negligibly thin, then $\vec{J}_i(\vec{x})\, d^3x = I_i\, d\vec{l}$

$$\Rightarrow \quad M_{ij} = \frac{1}{I_i I_j} \oint_{C_i} I_i\, d\vec{l} \cdot \vec{A}_j(\vec{x}) = \frac{1}{I_j} \int_S (\nabla \times \vec{A}_j) \cdot \hat{n}\, da = \frac{1}{I_j} \int_S \vec{B}_j \cdot d\vec{a}$$

= magnetic flux in *i* due to field produced by *j*, divided by current in *j*

Similarly for self-inductance.

$$emf = \oint_C \vec{E} \cdot d\vec{l} = -\frac{d}{dt} \int \vec{B} \cdot d\vec{a} = -\frac{d}{dt}(LI) = -L\frac{dI}{dt} \qquad \left(\text{or} \quad - M_{12}\frac{dI_2}{dt}\right)$$

SP 5.7, 5.8

Quasi-static Magnetic Fields in Conductors (Jackson sec 5.18)

Suppose the variation in $\vec{B}$ is sufficiently slow that $\vec{B}$ dominates over the induced $\vec{E}$

Conductor: $\vec{J} = \sigma \vec{E}$ (σ = conductivity)

Faraday's Law: $\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad \Rightarrow \quad \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$

$$\Rightarrow \quad \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \Phi \quad \Rightarrow \quad \vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t}$$

In cases of negligible free charge, the variation of $\vec{B}$ is the only source of $\vec{E}$

$$\Rightarrow \quad \vec{E} = -\frac{\partial \vec{A}}{\partial t}$$

Consider a medium with uniform and frequency-independent permeability μ and frequency-independent conductivity σ.

Ampère's Law: $\nabla \times \vec{B} = \mu \vec{J} = \mu\sigma\vec{E}$

$$\Rightarrow \quad \nabla \times (\nabla \times \vec{A}) = -\mu\sigma \frac{\partial \vec{A}}{\partial t}$$

Adopt Coulomb gauge: $\nabla \cdot \vec{A} = 0$

$$\Rightarrow \quad \nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A} = -\mu\sigma \frac{\partial \vec{A}}{\partial t}$$

$$\nabla^2 \vec{A} = \mu\sigma \frac{\partial \vec{A}}{\partial t}$$ (diffusion eqn)

Taking time derivative:

$$\nabla^2 \left(\frac{\partial \vec{A}}{\partial t} \right) = \mu\sigma \frac{\partial^2 \vec{A}}{\partial t^2} \quad \Rightarrow \quad \nabla^2 \vec{E} = \mu\sigma \frac{\partial \vec{E}}{\partial t}$$ (diffusion eqn again)

Suppose the field initially varies on a spatial scale $\sim L$

$$\Rightarrow \quad |\nabla^2 \vec{A}| \sim \frac{A}{L^2}$$

If the timescale for field decay is τ, then $\left|\frac{\partial \vec{A}}{\partial t}\right| \sim \frac{A}{\tau}$

$$\Rightarrow \quad \frac{A}{L^2} \sim \mu\sigma \frac{A}{\tau} \quad \Rightarrow \quad \tau \sim \mu\sigma L^2$$

Or, if the conductor is subjected to external fields that vary with frequency $\nu = \tau^{-1}$, then the fields penetrate into the conductor to a distance $L \sim (\mu\sigma\nu)^{-1/2}$