Maxwell's Equations and Conservation Laws

Reading: Jackson 6.1 through 6.4, 6.7

Ampère's Law, $\nabla \times \vec{H} = \vec{J} \;\Rightarrow\; \nabla \cdot \vec{J} = 0,$ since $\nabla \cdot (\nabla \times \vec{H}) = 0$ identically.

Although $\nabla \cdot \vec{J} = 0$ for magnetostatics, generally $\nabla \cdot \vec{J} + \frac{\partial \rho}{\partial t} = 0$

Maxwell suggested: Use Gauss's Law to rewrite continuity eqn:

$$\nabla \cdot \vec{J} + \frac{\partial}{\partial t}(\nabla \cdot \vec{D}) = 0 \quad \Rightarrow \quad \nabla \cdot \left(\vec{J} + \frac{\partial \vec{D}}{\partial t} \right) = 0$$

Thus, replace $\vec{J}$ with $\vec{J} + \frac{\partial \vec{D}}{\partial t}$ in Ampere's Law $\Rightarrow$ $\nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$

$\partial \vec{D} / \partial t$ is called the "displacement current".

Maxwell's Eqns:

$$\nabla \cdot \vec{D} = \rho \qquad \nabla \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

$$\nabla \cdot \vec{B} = 0 \qquad \nabla \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

Lorentz force law: $\vec{F} = q\,(\vec{E} + \vec{v} \times \vec{B})$

This set of 5 eqns, along with Newton's 2nd Law, provides a complete description of the classical dynamics of interacting charged particles and electromagnetic fields.

Vector and Scalar Potentials; Gauges (Jackson sec 6.2, 6.3)

$\nabla \cdot \vec{B} = 0$ is satisfied identically if we adopt the vector pot: $\boxed{\vec{B} = \nabla \times \vec{A}}$

Faraday's Law: $\nabla \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \quad \Rightarrow \quad \nabla \times \left(\vec{E} + \frac{\partial \vec{A}}{\partial t} \right) = 0$

This is satisfied identically if $\vec{E} + \frac{\partial \vec{A}}{\partial t} = -\nabla \Phi$, or, $\boxed{\vec{E} = -\nabla \Phi - \frac{\partial \vec{A}}{\partial t}}$

This reduces the number of equations from 4 to 2.

Recall that $\epsilon_0 \mu_0 = \frac{1}{c^2}$

In vacuum, the 2 inhomogeneous Maxwell eqns are

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0} \qquad (1)$$

$$\Rightarrow \quad \nabla \cdot \left(-\nabla\Phi - \frac{\partial \vec{A}}{\partial t} \right) = \frac{\rho}{\epsilon_0}$$

$$\Rightarrow \quad \boxed{\nabla^2\Phi + \frac{\partial}{\partial t}(\nabla \cdot \vec{A}) = -\frac{\rho}{\epsilon_0}}$$

(1b; generalization of Poisson's eqn)

$$\nabla \times \vec{B} = \mu_0 \vec{J} + \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t} \qquad (2)$$

$$\Rightarrow \quad \nabla \times (\nabla \times \vec{A}) = \mu_0 \vec{J} - \mu_0 \epsilon_0 \frac{\partial}{\partial t} \left(\nabla\Phi + \frac{\partial \vec{A}}{\partial t} \right)$$

Identity (in Jackson cover): $\nabla \times (\nabla \times \vec{A}) = \nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}$

$$\Rightarrow \quad \boxed{\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \nabla\left(\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right) = -\mu_0 \vec{J}} \qquad \text{(2b)}$$

Since $\vec{B} = \nabla \times \vec{A}$, $\vec{A} \rightarrow \vec{A}' = \vec{A} + \nabla\Lambda$ leaves $\vec{B}$ unchanged.

For $\vec{E}$ to remain unchanged as well, we require $\Phi \rightarrow \Phi' = \Phi - \dfrac{\partial \Lambda}{\partial t}$

$$\vec{E} = -\nabla\Phi - \frac{\partial \vec{A}}{\partial t}$$

$$\begin{aligned} -\nabla\Phi' - \frac{\partial \vec{A}'}{\partial t} &= -\nabla\left(\Phi - \frac{\partial \Lambda}{\partial t}\right) - \frac{\partial}{\partial t}\left(\vec{A} + \nabla\Lambda\right) \\ &= -\nabla\Phi + \frac{\partial}{\partial t}\nabla\Lambda - \frac{\partial \vec{A}}{\partial t} - \frac{\partial}{\partial t}\nabla\Lambda \\ &= \vec{E} \end{aligned}$$

$\vec{A} \rightarrow \vec{A} + \nabla\Lambda$, $\Phi \rightarrow \Phi - \dfrac{\partial \Lambda}{\partial t}$ is called a "gauge transformation".

Invariance of the fields under such transformations is called "gauge invariance" or "gauge freedom".

We can use gauge freedom to specify useful conditions on $\vec{A},\ \Phi$.

Lorenz gauge: $\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0$ ("Lorenz condition")

Suppose $\vec{A}, \Phi$ do not satisfy the Lorenz condition. We seek a scalar field Λ such that gauge-transformed potentials do:

$$\nabla \cdot (\vec{A} + \nabla\Lambda) + \frac{1}{c^2}\frac{\partial}{\partial t}\left(\Phi - \frac{\partial \Lambda}{\partial t}\right) = 0$$

$$\Rightarrow \quad \nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = -\left(\nabla \cdot \vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right) \qquad (3)$$

This is the wave eqn for Λ. Since the wave eqn has a solution for any source, we can always find a Λ that will yield potentials satisfying the Lorenz condition.

Applying the Lorenz condition to (1b) and (2b):

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \qquad \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J}$$

Thus, we have 2 uncoupled wave eqns for Φ and $\vec{A}$. These are much easier to solve than the coupled eqns. But, we must only employ solns that satisfy the Lorenz condition!

The Lorenz condition does not uniquely specify the potentials. If $\vec{A}$, Φ satisfy it, then so do $\vec{A}' = \vec{A} + \nabla\Lambda$, $\Phi' = \Phi - \frac{\partial \Lambda}{\partial t}$

if $\nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = 0$ (from eq 3)

Coulomb gauge: $\nabla \cdot \vec{A} = 0$ (also called "radiation" or "transverse" gauge)

In this case, (1b) => $\nabla^2\Phi = -\frac{\rho}{\epsilon_0}$, identical to Poisson's eqn of electrostatics

$$\Rightarrow \quad \Phi(\vec{x}, t) = \frac{1}{4\pi\epsilon_0}\int \frac{\rho(\vec{x}', t)}{|\vec{x} - \vec{x}'|} d^3x'$$ ("instantaneous Coulomb potential")

$$(2b) \quad \Rightarrow \quad \nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} + \frac{1}{c^2}\nabla\frac{\partial\Phi}{\partial t} \qquad (4)$$

We will make use of Helmholtz's Theorem: Any vector field $\vec{J}$ whose divergence and curl vanish at infinity can be expressed as the sum of a "longitudinal", or "irrotational", term $\vec{J}_l$ with $\nabla \times \vec{J}_l = 0$ and a "transverse", or "solenoidal", term $\vec{J}_t$ with $\nabla \cdot \vec{J}_t = 0$.

$\vec{J} = \vec{J}_l + \vec{J}_t$ with

SP 6.1

$$\vec{J}_l(\vec{x},t) = -\frac{1}{4\pi}\nabla_x \int \frac{\nabla_{x'} \cdot \vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x' \qquad \vec{J}_t(\vec{x},t) = \frac{1}{4\pi}\nabla_x \times \nabla_x \times \int \frac{\vec{J}(\vec{x}')}{|\vec{x}-\vec{x}'|} d^3x'$$

$$\begin{aligned}
\frac{1}{c^2}\nabla\frac{\partial\Phi}{\partial t} &= \mu_0\epsilon_0 \frac{1}{4\pi\epsilon_0}\nabla_x \int \frac{\partial\rho(\vec{x}',t)}{\partial t}\frac{1}{|\vec{x}-\vec{x}'|} d^3x' \\
&= \mu_0\left(-\frac{1}{4\pi}\right)\nabla_x \int \frac{\nabla_{x'} \cdot \vec{J}(\vec{x}',t)}{|\vec{x}-\vec{x}'|} d^3x' \qquad \text{(using the continuity eqn)} \\
&= \mu_0 \vec{J}_l
\end{aligned}$$

Then, (4) becomes
$$\nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} = -\mu_0\vec{J} + \mu_0\vec{J}_l = -\mu_0\vec{J}_t$$

Note: In Coulomb gauge, Φ propagates instantaneously. That is, a change in ρ at $\vec{x}'$ results in a change in Φ at $\vec{x}$ without any delay. But, $\vec{A}$ propagates at the speed of light. Φ and $\vec{A}$ together determine $\vec{E}$ and $\vec{B}$ in such a way that the fields are modified according to a speed-of-light delay.

Green Functions for the Wave Equation (Jackson sec 6.4)

In both the Lorenz and Coulomb gauges, we find wave eqns for the potentials. The wave eqn has the form:

$$\nabla^2\Psi(\vec{x},t) - \frac{1}{c^2}\frac{\partial^2\Psi(\vec{x},t)}{\partial t^2} = -4\pi f(\vec{x},t)$$

For a simple demonstration that the solns are waves, consider a scalar field Ψ in 1-D and no source term (i.e., $f = 0$; homogeneous eqn).

$$\nabla^2\Psi = \frac{\partial^2\Psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2}$$

A wave with speed *c* has the form $\Psi(x,t) = g(z \equiv x \pm ct)$
(- for waves traveling in the +*x* direction, + for -*x* direction)

$$\frac{\partial \Psi}{\partial x} = \frac{dg}{dz}\frac{\partial z}{\partial x} = \frac{dg}{dz}$$

$$\frac{\partial^2 \Psi}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{dg}{dz}\right) = \frac{d^2 g}{dz^2}\frac{\partial z}{\partial x} = \frac{d^2 g}{dz^2}$$

$$\frac{\partial \Psi}{\partial t} = \frac{dg}{dz}\frac{\partial z}{\partial t} = (\pm c)\frac{dg}{dz}$$

$$\frac{\partial^2 \Psi}{\partial t^2} = (\pm c)\frac{\partial}{\partial t}\left(\frac{dg}{dz}\right) = (\pm c)\frac{d^2 g}{dz^2}\frac{\partial z}{\partial t} = (\pm c)(\pm c)\frac{d^2 g}{dz^2} = c^2\frac{d^2 g}{dz^2}$$

$$\Rightarrow \quad \frac{\partial^2 \Psi}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2}$$

=> the wave is indeed a soln

For the 3-D problem with non-zero source, we'll use a Green fcn method. Assume that the region of interest is all space => no bounding surfaces.

Recall the use of the Green function in electrostatics:

$$\nabla^2\Phi = -\frac{\rho}{\epsilon_0} = -4\pi\, f(\vec{x}) \qquad \text{with} \qquad f(\vec{x}) = \frac{\rho(\vec{x})}{4\pi\epsilon_0}$$

Introduce Green function $G(\vec{x},\vec{x}')$ such that $\nabla^2_{x'} G(\vec{x},\vec{x}') = -4\pi\,\delta(\vec{x}-\vec{x}')$

[can also differentiate wrt $\vec{x}$, since $G(\vec{x},\vec{x}') = G(\vec{x}',\vec{x})$]

Soln for all space: $G(\vec{x},\vec{x}') = \dfrac{1}{|\vec{x}-\vec{x}'|}$

For all space, Green's Thm yields

$$\Phi(\vec{x}) = \int f(\vec{x}')\, G(\vec{x},\vec{x}')\, d^3x' = \frac{1}{4\pi\epsilon_0}\int \rho(\vec{x}')\, G(\vec{x},\vec{x}')\, d^3x'$$

Check: $\nabla^2\Phi = \int f(\vec{x}')\,\nabla_x^2 G(\vec{x},\vec{x}')\,d^3x' = \int f(\vec{x}')\,[-4\pi\delta(\vec{x}-\vec{x}')]\,d^3x' = -4\pi\,f(\vec{x})$

Now, we have $\nabla^2\Psi - \frac{1}{c^2}\frac{\partial^2\Psi}{\partial t^2} = -4\pi\,f(\vec{x},t)$

Introduce a Green fcn satisfying

$$\left(\nabla_x^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{x},t;\vec{x}',t') = -4\pi\,\delta(\vec{x}-\vec{x}')\,\delta(t-t') \qquad (1)$$

Then, $$\Psi(\vec{x},t) = \int f(\vec{x}',t')\,G(\vec{x},t;\vec{x}',t')\,d^3x'\,dt' \qquad (2)$$

Check:

$$\left(\nabla_x^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right)\Psi = \int f(\vec{x}',t')\,[-4\pi\,\delta(\vec{x}-\vec{x}')\,\delta(t-t')]\,d^3x'\,dt' = -4\pi\,f(\vec{x},t)$$

The physical interpretation of this Green fcn is odd: delta-fcn source in both space and time => source that flits into existence just for an instant, then ceases to exist

First, use a Fourier transform (Jackson 2.44 and 2.45) to eliminate the explicit time dependence:

$$G(\vec{x}, t; \vec{x}', t') = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(\vec{x}, \omega; \vec{x}', t')\, e^{-i\omega t}\, d\omega \qquad (3)$$

Note the common, but potentially confusing, practice of using the same variable, with different arguments, for the fcn and Fourier coeffs.

Also, recall that $\delta(t - t') = \dfrac{1}{2\pi} \displaystyle\int_{-\infty}^{\infty} e^{-i\omega(t-t')}\, d\omega$ (Jackson 2.47)

The wave eqn for G becomes:

$$\frac{1}{2\pi} \int_{-\infty}^{\infty} \left[\nabla_x^2 G(\vec{x}, \omega; \vec{x}', t')\, e^{-i\omega t} + \frac{\omega^2}{c^2} G(\vec{x}, \omega; \vec{x}', t')\, e^{-i\omega t} \right] d\omega = -4\pi\, \delta(\vec{x} - \vec{x}')\, \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-i\omega(t-t')}\, d\omega$$

This must be true for all t' =>

$$\nabla_x^2 G(\vec{x}, \omega; \vec{x}', t') + \frac{\omega^2}{c^2} G(\vec{x}, \omega; \vec{x}', t') = -4\pi\, \delta(\vec{x} - \vec{x}')\, e^{i\omega t'} \qquad (4)$$

$$\nabla_x^2 G(\vec{x},\omega;\vec{x}',t') + \frac{\omega^2}{c^2} G(\vec{x},\omega;\vec{x}',t') = -4\pi\,\delta(\vec{x}-\vec{x}')\,e^{i\omega t'} \qquad (4)$$

This reduces to the eqn for the electrostatic Green fcn when $\omega = 0$.

Since we are considering all space, we may place the origin anywhere (e.g., at $\vec{x}'$) => G depends only on $\vec{R} \equiv \vec{x} - \vec{x}'$, not on $\vec{x}$ and $\vec{x}'$ separately. We also have complete freedom re. orientation of axes => G depends only on $R = |\vec{R}|$.

$$\Rightarrow \quad G(\vec{x},\omega;\vec{x}',t') = G(R)\,e^{i\omega t'} \qquad (5)$$

Adopting spherical coords, eqn (4) for $G(R)$ becomes

$$\frac{1}{R}\frac{d^2}{dR^2}(RG) + \frac{\omega^2}{c^2}G = -4\pi\,\delta(\vec{R})$$

Everywhere but $R = 0$, $\quad \frac{d^2}{dR^2}(RG) + \frac{\omega^2}{c^2}(RG) = 0$

$$\Rightarrow \quad R\,G(R) = A\,e^{i\omega R/c} + B\,e^{-i\omega R/c}$$

$$\Rightarrow \quad G(R) = A\,\frac{e^{i\omega R/c}}{R} + B\,\frac{e^{-i\omega R/c}}{R}$$

In the limit $\omega \to 0$, this must yield the electrostatic Green fcn for all space:

$$G = \frac{1}{|\vec{x} - \vec{x}'|} = \frac{1}{R}$$

$$\Rightarrow \quad \lim_{\omega \to 0}\left[A\,\frac{e^{i\omega R/c}}{R} + B\,\frac{e^{-i\omega R/c}}{R}\right] = \frac{1}{R} \qquad\qquad \Rightarrow \quad A + B = 1$$

Thus, $G(R) = A\,G^{(+)}(R) + (1 - A)\,G^{(-)}(R)$

with $\quad G^{(\pm)}(R) = \dfrac{e^{\pm i\omega R/c}}{R} \qquad\qquad (6)$

These are outgoing (+) and incoming (-) spherical waves. The value of *A* is determined by the time boundary condition.

From eqns (3), (5), and (6):

$$
\begin{aligned}
G^{(\pm)}(\vec{x}, t; \vec{x}', t') &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{e^{\pm i\omega R/c}}{R}\, e^{i\omega t'}\, e^{-i\omega t}\, d\omega \\
&= \frac{1}{R}\,\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega[t'-(t\mp R/c)]}\, d\omega \\
&= \frac{\delta\left[t' - \left(t \mp \frac{R}{c}\right)\right]}{R} \\
&= \frac{\delta\left[t' - \left(t \mp \frac{|\vec{x}-\vec{x}'|}{c}\right)\right]}{|\vec{x}-\vec{x}'|}
\end{aligned}
$$

$G^{(+)}$ is the "retarded Green function" and $G^{(-)}$ is the "advanced Green function". For $G^{(+)}$, a disturbance occurs at $(\vec{x}', t')$ and the news reaches $\vec{x}$ at a later time *t*, with the delay $\Delta t = R/c$. The news travels at speed *c*.

From eqn (2), solns of the wave eqn are:

$$
\Psi^{(\pm)}(\vec{x}, t) = \int f(\vec{x}', t')\, G^{(\pm)}(\vec{x}, t; \vec{x}', t')\, d^3x'\, dt'
$$

To match boundary conditions, solns of the homogeneous wave eqn are added.

A simple, common situation is that $\Psi \to 0$ as $t \to -\infty$, in which case the retarded soln $\Psi^{(+)}(\vec{x},t)$ applies.

$$\Rightarrow \quad \Psi(\vec{x},t) = \int f(\vec{x}',t')\,\frac{\delta[t'-(t-|\vec{x}-\vec{x}'|/c)]}{|\vec{x}-\vec{x}'|}\,d^3x'\,dt'$$

$$= \int \frac{[f(\vec{x}',t')]_{\rm ret}}{|\vec{x}-\vec{x}'|}\,d^3x'$$

where $[...]_{\rm ret}$ indicates that $t' = t - |\vec{x}-\vec{x}'|/c$, the retarded time.

Recall the wave eqn for the potentials in the Lorenz gauge:

$$\nabla^2\Phi - \frac{1}{c^2}\frac{\partial^2\Phi}{\partial t^2} = -\frac{\rho}{\epsilon_0} \quad \Rightarrow \quad f = \frac{\rho}{4\pi\epsilon_0}$$

$$\nabla^2\vec{A} - \frac{1}{c^2}\frac{\partial^2\vec{A}}{\partial t^2} = -\mu_0\vec{J} \quad \Rightarrow \quad f = \frac{\mu_0\vec{J}}{4\pi}$$

$$\Rightarrow \quad \Phi(\vec{x},t) = \frac{1}{4\pi\epsilon_0}\int \frac{[\rho(\vec{x}',t')]_{\rm ret}}{|\vec{x}-\vec{x}'|}d^3x' \qquad \vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\int \frac{[\vec{J}(\vec{x}',t')]_{\rm ret}}{|\vec{x}-\vec{x}'|}d^3x'$$

In a sample problem, we'll verify that these potentials satisfy the Lorenz condition.

Suppose the source is a moving point charge q with trajectory $\vec{r}(t)$; velocity $\vec{v}(t) = d\vec{r}/dt$.

$$\rho(\vec{x}',t') = q\,\delta[\vec{x}' - \vec{r}(t')] \qquad \vec{J}(\vec{x}',t') = q\vec{v}(t')\,\delta[\vec{x}' - \vec{r}(t')]$$

$$\Phi(\vec{x},t) = \frac{q}{4\pi\epsilon_0}\int \frac{\{\delta\,[\vec{x}' - \vec{r}(t')]\}_{\rm ret}}{|\vec{x}-\vec{x}'|}\,d^3x'$$

$$= \frac{q}{4\pi\epsilon_0}\int \frac{\delta\,[\vec{x}' - \vec{r}\,(t - |\vec{x}-\vec{x}'|/c)]}{|\vec{x}-\vec{x}'|}\,d^3x'$$

$$= \frac{q}{4\pi\epsilon_0}\int dt'\,d^3x'\,\delta\left(t' - t + \frac{|\vec{x}-\vec{x}'|}{c}\right)\frac{\delta\,[\vec{x}' - \vec{r}(t')]}{|\vec{x}-\vec{x}'|}$$

$$\Phi(\vec{x},t) = \frac{q}{4\pi\epsilon_0}\int dt'\,d^3x'\,\delta\left(t'-t+\frac{|\vec{x}-\vec{x}'|}{c}\right)\frac{\delta\left[\vec{x}'-\vec{r}(t')\right]}{|\vec{x}-\vec{x}'|}$$

$$= \frac{q}{4\pi\epsilon_0}\int dt'\,\delta\left(t'-t+\frac{|\vec{x}-\vec{r}(t')|}{c}\right)\frac{1}{|\vec{x}-\vec{r}(t')|}$$

Substitute $u = t'-t+\frac{|\vec{x}-\vec{r}(t')|}{c} = t'-t+\frac{\sqrt{x^2+r^2(t')-2\,\vec{x}\cdot\vec{r}(t')}}{c}$

$$du = dt' + \frac{2r\frac{dr}{dt'} - 2\,\vec{x}\cdot\vec{v}(t')}{2c\,|\vec{x}-\vec{r}(t')|}\,dt'$$

Note: $\frac{d(r^2)}{dt'} = 2r\frac{dr}{dt'}$ Also, $\frac{d(r^2)}{dt'} = \frac{d(\vec{r}\cdot\vec{r})}{dt'} = 2\vec{r}\cdot\frac{d\vec{r}}{dt'} = 2\vec{r}\cdot\vec{v}$

$\Rightarrow \quad r\,\frac{dr}{dt'} = \vec{r}\cdot\vec{v} \qquad \Rightarrow \quad du = dt'\left\{1+\frac{[\vec{r}(t')-\vec{x}]}{c\,|\vec{x}-\vec{r}(t')|}\cdot\vec{v}(t')\right\}$

$$\Rightarrow \quad \Phi(\vec{x},t) = \frac{q}{4\pi\epsilon_0}\int \frac{du\,\delta(u)}{|\vec{x}-\vec{r}(t')|\,\left\{1 - \frac{[\vec{x}-\vec{r}(t')]}{|\vec{x}-\vec{r}(t')|}\cdot\frac{\vec{v}(t')}{c}\right\}}$$

$$= \frac{q}{4\pi\epsilon_0}\;\frac{1}{|\vec{x}-\vec{r}(t')|\,\left\{1 - \frac{[\vec{x}-\vec{r}(t')]}{|\vec{x}-\vec{r}(t')|}\cdot\frac{\vec{v}(t')}{c}\right\}}\Bigg|_{t'=t-|\vec{x}-\vec{r}(t')|/c}$$

Define $\hat{n} =$ a unit vector pointing in the direction $\vec{x}-\vec{r}(t')$.

$$R = |\vec{x}-\vec{r}(t')| \qquad , \qquad \vec{\beta} = \frac{\vec{v}(t')}{c}$$

$$\Phi(\vec{x},t) = \left[\frac{q}{4\pi\epsilon_0 R\,(1-\vec{\beta}\cdot\hat{n})}\right]_{\text{ret}}$$

where we evaluate R, $\vec{\beta}$, $\hat{n}$ at the retarded time t', which is defined implicitly by $t' = t - R/c$, or, $R = c\,(t - t')$.

The calculation is nearly identical for $\vec{A}$:

$$\vec{A}(\vec{x},t) = \frac{\mu_0}{4\pi}\left[\frac{q\,\vec{\beta}\,c}{R\,(1-\vec{\beta}\cdot\hat{n})}\right]_{\text{ret}} \qquad = \left[\frac{q\,\vec{\beta}}{4\pi\epsilon_0 cR\,(1-\vec{\beta}\cdot\hat{n})}\right]_{\text{ret}}$$

$$= \frac{\vec{\beta}_{\text{ret}}}{c}\,\Phi(\vec{x},t)$$

These are known as the "Liénard-Wiechert potentials". **SP 6.2, 6.3**

Conservation of Energy and Momentum (Jackson sec 6.7)

Power done by external fields on a charge q is $\vec{F}\cdot\vec{v} = q\,\vec{v}\cdot\vec{E}$

$$\Rightarrow \quad dW/dt = \vec{J}\cdot\vec{E}\,d^3x$$

$$\vec{J}\cdot\vec{E} = \left[(\nabla\times\vec{H}) - \frac{\partial\vec{D}}{\partial t}\right]\cdot\vec{E} = \vec{E}\cdot(\nabla\times\vec{H}) - \vec{E}\cdot\frac{\partial\vec{D}}{\partial t}$$

Vector identity: $\nabla \cdot (\vec{E} \times \vec{H}) = \vec{H} \cdot (\nabla \times \vec{E}) - \vec{E} \cdot (\nabla \times \vec{H})$

$$\Rightarrow \quad \vec{J} \cdot \vec{E} = -\left[\nabla \cdot (\vec{E} \times \vec{H}) - \vec{H} \cdot \underbrace{(\nabla \times \vec{E})}_{= -\partial \vec{B}/\partial t} + \vec{E} \cdot \frac{\partial \vec{D}}{\partial t}\right]$$

Suppose medium is linear with negligible dispersion or losses

$$\Rightarrow \quad \vec{D} = \epsilon \vec{E} \quad \text{and} \quad \vec{B} = \mu \vec{H}$$

ϵ and μ are real and frequency-independent

Define $u = \frac{1}{2}\left(\vec{E} \cdot \vec{D} + \vec{B} \cdot \vec{H}\right) = \frac{1}{2}\left(\epsilon \vec{E} \cdot \vec{E} + \mu \vec{H} \cdot \vec{H}\right)$

$$\frac{\partial u}{\partial t} = \epsilon \vec{E} \cdot \frac{\partial \vec{E}}{\partial t} + \mu \vec{H} \cdot \frac{\partial \vec{H}}{\partial t} = \vec{E} \cdot \frac{\partial \vec{D}}{\partial t} + \vec{H} \cdot \frac{\partial \vec{B}}{\partial t}$$

Also, define $\vec{S} = \vec{E} \times \vec{H} \quad \Rightarrow \quad \vec{J} \cdot \vec{E} = -\left[\nabla \cdot \vec{S} + \frac{\partial u}{\partial t}\right]$

$$\Rightarrow \quad \nabla \cdot \vec{S} + \frac{\partial u}{\partial t} = -\vec{J} \cdot \vec{E} \qquad (1)$$

RHS = negative of rate per unit volume at which fields do work on particles = rate at which field energy increases per unit volume

=> u = field energy density

$\vec{S}$ = field energy flux density vector ("Poynting vector")

(1) expresses conservation of (mechanical plus electromagnetic field) energy.

To develop conservation of momentum, start with the electromagnetic force on a charge q : $\vec{F} = q(\vec{E} + \vec{v} \times \vec{B})$

If the sum of the momenta of all the particles in volume V is $\vec{P}_{\text{mech}}$, then

$$\frac{d\vec{P}_{\text{mech}}}{dt} = \int_V (\rho\vec{E} + \vec{J} \times \vec{B})\, d^3x$$

$$\rho\vec{E} + \vec{J}\times\vec{B} = \left[\epsilon_0\nabla\cdot\vec{E}\right]\vec{E} + \left[\frac{1}{\mu_0}\nabla\times\vec{B} - \epsilon_0\frac{\partial\vec{E}}{\partial t}\right]\times\vec{B}$$

$$= \epsilon_0\left[\vec{E}\,\nabla\cdot\vec{E} + \vec{B}\times\frac{\partial\vec{E}}{\partial t} - c^2\,\vec{B}\times(\nabla\times\vec{B})\right]$$

$$\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) = \vec{E}\times\frac{\partial\vec{B}}{\partial t} + \frac{\partial\vec{E}}{\partial t}\times\vec{B}$$

$$\Rightarrow\quad \vec{B}\times\frac{\partial\vec{E}}{\partial t} = -\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) + \vec{E}\times\frac{\partial\vec{B}}{\partial t} = -\frac{\partial}{\partial t}(\vec{E}\times\vec{B}) - \vec{E}\times(\nabla\times\vec{E})$$

$$\rho\vec{E} + \vec{J}\times\vec{B} = \epsilon_0\left[\vec{E}\,(\nabla\cdot\vec{E}) + c^2\,\vec{B}\,(\nabla\cdot\vec{B}) - \vec{E}\times(\nabla\times\vec{E}) - c^2\,\vec{B}\times(\nabla\times\vec{B})\right] - \epsilon_0\frac{\partial}{\partial t}(\vec{E}\times\vec{B})$$

= 0, so we're adding zero here

$$\Rightarrow\quad \frac{d\vec{P}_{\mathrm{mech}}}{dt} + \frac{d}{dt}\int_V \epsilon_0\,(\vec{E}\times\vec{B})\,d^3x =$$

$$\epsilon_0\int_V\left[\vec{E}\,(\nabla\cdot\vec{E}) - \vec{E}\times(\nabla\times\vec{E}) + c^2\,\vec{B}\,(\nabla\cdot\vec{B}) - c^2\,\vec{B}\times(\nabla\times\vec{B})\right]d^3x \qquad (2)$$

Consider the term $\vec{E}\,(\nabla\cdot\vec{E}) - \vec{E}\times(\nabla\times\vec{E})$ on the RHS.

If the Cartesian coords are x_α (α = 1, 2, 3), then the $\alpha = 1$ component is

$$E_1\left(\frac{\partial E_1}{\partial x_1}+\frac{\partial E_2}{\partial x_2}+\frac{\partial E_3}{\partial x_3}\right) - E_2\left(\frac{\partial E_2}{\partial x_1}-\frac{\partial E_1}{\partial x_2}\right) + E_3\left(\frac{\partial E_1}{\partial x_3}-\frac{\partial E_3}{\partial x_1}\right)$$

$$= \frac{\partial}{\partial x_1}(E_1^2) + \frac{\partial}{\partial x_2}(E_1\,E_2) + \frac{\partial}{\partial x_3}(E_1\,E_3) - \frac{1}{2}\,\frac{\partial}{\partial x_1}(E_1^2+E_2^2+E_3^2)$$

$$= \sum_{\beta=1}^{3}\frac{\partial}{\partial x_\beta}\left(E_1\,E_\beta - \frac{1}{2}\,\vec{E}\cdot\vec{E}\,\delta_{1\beta}\right)$$

$$\Rightarrow \quad \left[\vec{E}\,(\nabla\cdot\vec{E}) - \vec{E}\times(\nabla\times\vec{E})\right]_\alpha = \sum_{\beta=1}^{3}\frac{\partial}{\partial x_\beta}\left(E_\alpha\,E_\beta - \frac{1}{2}\,\vec{E}\cdot\vec{E}\,\delta_{\alpha\beta}\right)$$

=> α-component of RHS of (2) $= \int_V \sum_{\beta=1}^{3}\frac{\partial}{\partial x_\beta}T_{\alpha\beta}\,d^3x$

with $T_{\alpha\beta} = \epsilon_0\left[E_\alpha\,E_\beta + c^2\,B_\alpha\,B_\beta - \frac{1}{2}\left(\vec{E}\cdot\vec{E} + c^2\,\vec{B}\cdot\vec{B}\right)\delta_{\alpha\beta}\right]$

The integrand is the divergence of $T_{\alpha\beta}$:

$$\int_V \sum_{\beta=1}^{3} \frac{\partial}{\partial x_\beta} T_{\alpha\beta}\, d^3x = \oint_S \sum_{\beta=1}^{3} T_{\alpha\beta}\, n_\beta\, da$$

(n_β = β-component of unit normal $\hat{n}$)

(2) becomes

$$\left[\frac{d\vec{P}_{\text{mech}}}{dt} + \frac{d}{dt}\int_V \epsilon_0\,(\vec{E}\times\vec{B})\,d^3x\right]_\alpha = \oint_S \sum_{\beta=1}^{3} T_{\alpha\beta}\, n_\beta\, da$$

=> electromagnetic field momentum $\vec{P}_{\text{field}} = \int_V \epsilon_0\,(\vec{E}\times\vec{B})\,d^3x$

and $\oint_S \sum_{\beta=1}^{3} T_{\alpha\beta}\, n_\beta\, da$ = the flux of α-component of field momentum into volume V

=> the field momentum density $\vec{g} = \epsilon_0\,\vec{E}\times\vec{B} = \frac{1}{c^2}\,\vec{E}\times\vec{H} = \frac{1}{c^2}\,\vec{S}$

"Maxwell stress tensor" $T_{\alpha\beta}$ = rate at which α-component of momentum crosses unit area along the -β direction.

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