Modified gravity and the origin of inertia

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Modified gravity theory is known to violate Birkhoff’s theorem. We explore a key consequence of this violation, the effect of distant matter in an Einstein-de Sitter universe on the motion of test particles. We find that when a particle is accelerated, a force is experienced that is proportional to the particle’s mass and acceleration and acts in the direction opposite to that of the acceleration. We identify this force with inertia. At very low accelerations, our inertial law deviates slightly from that of Newton, yielding a testable prediction that may be verified with relatively simple experiments.

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Modified Gravity (MOG) \textsuperscript{1} is a fully relativistic theory of gravitation that is based on postulating the existence of a massive vector field, $\phi_{\mu}$. The choice of a massive vector field is motivated by our desire to introduce a repulsive modification of the law of gravitation at short range. The vector field is coupled universally to matter. The theory, therefore, has three constants: in addition to the gravitational constant $G$, we must also consider the coupling constant $\omega$ that determines the coupling strength between the $\phi_{\mu}$ field and matter, and a further constant $\mu$ that arises as a result of considering a vector field of non-zero mass, and controls the coupling range. In the most general case, these constants must be allowed to run with distance (energy).

General relativity satisfies Birkhoff’s theorem \textsuperscript{2}. The metric inside an empty spherical cavity in the center of a spherically symmetric system is the Minkowski metric \textsuperscript{3}. This is one reason why attempts such as that of Scrima \textsuperscript{4} remain unconvincing, as also demonstrated (without referring to Birkhoff’s theorem) by Brans \textsuperscript{5}.

In contrast, MOG is known to violate Birkhoff’s theorem. Inside a spherically symmetric shell of matter, the MOG force is non-zero.

We investigate MOG in the weak-field approximation, in the case of a spherically symmetric, homogeneous shell of density $\rho$, inner radius $R_1$ and outer radius $R_2$, with a test particle of mass $m$ located at distance $z$ from the shell’s center. We parameterize the shell using spherical coordinates $r$, $\theta$, and $\phi$, where $\theta$ is the angle between the line connecting a point in the shell with the center of the shell, and the line connecting the test particle to the center of the shell, and $\phi$ the angle of rotation in a plane perpendicular to the line connecting the test particle with the center of the shell, relative to a preferred direction.

The distance $l$ between a point in the shell and the test particle can be written as

$$l^2 = z^2 + r^2 - 2rz \cos \theta. \quad (1)$$

In the Newtonian case, the gravitational force obeys the inverse square law. The gravitational force on the test particle can therefore be written as

$$F = \int \frac{G\rho m(r-z)}{l^3} \, dV, \quad (2)$$

where $G = G_N$ is Newton’s constant, $dV$ denotes a volume element inside the shell, and the integration is carried out for the entire volume of the shell. The volume element can be expressed using coordinates as

$$dV = r^2 \sin \theta \, dr \, d\theta \, d\phi. \quad (3)$$

Because of spherical symmetry, components of $F$ perpendicular to the line connecting the test particle and the center of the shell vanish. The component parallel to this line, which we denote with $F_0$, can be calculated using the projection coefficient $(z-r \cos \theta)/l$:

$$F_0 = \int_0^{2\pi} \int_0^{\pi} \int_{R_1}^{R_2} G_N \rho m r^2 (z-r \cos \theta) \sin \theta \frac{dr \, d\theta \, d\phi}{l^3}. \quad (4)$$

This integral evaluates to

$$F_0 = \begin{cases} 0 & z < R_1 \\ 4\pi G_N \rho m (z^3 - R_1^3)/3z^2 & R_1 \leq z < R_2 \\ 4\pi G_N \rho m (R_2^3 - R_1^3)/3z^2 & R_2 \leq z. \end{cases} \quad (5)$$

In the weak-field approximation, the MOG acceleration law can be written as Newton’s law of gravity with an effective gravitational constant that incorporates a repulsive Yukawa term\textsuperscript{1}:

$$G = G_\infty \left[ 1 - \frac{\alpha}{1 + \alpha} (1 + \mu l) e^{-\mu l} \right]. \quad (6)$$

In the case of a spherical volume of uniform mass density and radius $R$, after evaluating (2), the gravitational

\textsuperscript{1} Setting $G_\infty = (1+\alpha)G_N$, where $G_N$ denotes Newton’s constant, we recover Newton’s law of gravity in the limit $l \to 0$, as shown in Ref. \textsuperscript{1}.
force in the interior and the exterior, denoted by \( F_1 \) and \( F_2 \), respectively, are written as

\[
F_1(R, z) = \pi G_\infty \rho m \left( \frac{4z}{3} - \frac{2\alpha}{1 + \alpha} \right)
\times \frac{\mu R + 1}{\mu^3 z^2} \left[ (\mu z + 1)e^{-\mu(R-z)} + (\mu z - 1)e^{-\mu(R+z)} \right],
\]

\[
F_2(R, z) = \pi G_\infty \rho m \left( \frac{4R^3}{3z^2} - \frac{2\alpha}{1 + \alpha} \right)
\times \frac{\mu R + 1}{\mu^3 z^2} \left[ (\mu R + 1)e^{-\mu(R+z)} + (\mu R - 1)e^{-\mu(R-z)} \right].
\]

(7)

The gravitational force inside, within, and outside a spherical shell of inner radius \( R_1 \) and outer radius \( R_2 \) can be written as

\[
F_0 = \begin{cases} 
F_1(R_2, z) - F_1(R_1, z) & z < R_1 \vs 
F_1(R_2, z) - F_1(R_1, z) & R_1 \leq z < R_2 \vs 
F_2(R_2, z) - F_2(R_1, z) & R_2 \leq z.
\end{cases}
\]

(9)

In particular, \( F_1(R_2, z) - F_1(R_1, z) \) is nonvanishing; the net force in the interior of a spherically symmetric shell is not zero.

For an infinitesimally thin shell, we can write:

\[
dF = \lim_{R_2 \to R_1} \left[ F_1(R_2, z) - F_1(R_1, z) \right],
\]

(10)

or

\[
\mathcal{F}(R, z) = \frac{dF}{dR} = \lim_{R_2 \to R_1} \frac{F_1(R_2, z) - F_1(R_1, z)}{R_2 - R_1} \bigg|_{R_1 = R}.
\]

(11)

We imagine a test particle that is surrounded by an infinite series of infinitesimally thin concentric shells of matter. If the test particle moves with velocity \( v \), these shells are dragged along. However, because of the finite propagation velocity of the gravitational interaction, they will be delayed, so they will no longer be concentric. Therefore, a force will act on the particle. This force can be calculated by integrating the displacement of the particle relative to each of the concentric shells:

\[
F(z, v) = \int_0^\infty \mathcal{F}(R, z - Rv/c) \, dR.
\]

(12)

When a particle is moving with constant velocity along a straight line, it should feel no net force. This is achieved by assuming that after the particle has been set into motion, a momentary force acting on it displaced it, so it is no longer where it used to be relative to its shells of matter. The displacement can be calculated by solving the equation

\[
F(z, v) = 0
\]

for \( z \). This equation is difficult to solve exactly, but for nonrelativistic velocities, it yields

\[
z \simeq \frac{2}{\mu c} v,
\]

(14)

or

\[
\frac{dz}{dt} \simeq \frac{2}{\mu c} \frac{dv}{dt} = \frac{2}{\mu c} g.
\]

(15)

The value \( dz/dt \) is a velocity, specifically the velocity with which the particle is trying to move towards the new equilibrium position as a result of the acceleration \( g \). Unfortunately, Eq. (15) is nonrelativistic.

In the cosmological context, the coefficient \( \mu \) is set to the reciprocal of the horizon scale, yielding good agreement with key cosmological observations [6]. In this case, \( g_0 = \mu c = cH_0 \simeq 7 \times 10^{-10} \text{ m/s}^2 \). Eq. (15) is valid only for values \( g \ll g_0 \).

As we are working in the weak-field approximation, we can only address this deficiency phenomenologically. A “quick fix” is to introduce a relativistic correction to Eq. (15), in the form

\[
\frac{dz}{dt} \simeq \left( 1 + \frac{1}{c^2} \frac{2}{\mu c} g \right)^{-1} \frac{2}{\mu c} g.
\]

(16)

We can substitute this result back into Eq. (12), to get

\[
F(g) \simeq \pi G_\infty \rho m \frac{\alpha}{2 \mu^2 c^2} \left[ \frac{4g}{\mu c} \right] \sqrt{\log \left( 1 + \frac{4g}{\mu c} \right) - 4g \frac{\mu c^2 + 2g}{\mu c^2 + 4g}}
\]

\[
\times \left( \mu c^2 + 2g \right)^2 \frac{\mu^2 c^2 + 4g}{9 g^2}.
\]

(17)

When \( g \gg \mu c^2 \), this expression can be simplified. In particular, the linear term inside the square brackets will dominate over the logarithmic term, which can therefore be omitted. We are left with

\[
F(g) \simeq -\frac{\pi G_\infty \rho}{2 \mu^2 c^2} \frac{\alpha}{1 + \alpha} \frac{4g}{g_{\mu c^2} + 4g} \simeq -\frac{4 \pi G_\infty \rho}{3 \mu^2 c^2} \frac{\alpha}{1 + \alpha} mg.
\]

(18)

Note that when we set \( \mu = 1/a_0 \), then \( \mu c = H_0 \) as stated earlier. Furthermore, in our cosmological model,

\[
\frac{8 \pi G_\infty \rho}{3 H_0^2} = 1
\]

(19)
by definition [6], yielding an Einstein-de Sitter cosmology. Therefore,

$$F(g) \simeq -\frac{3}{2}\frac{\alpha}{1+\alpha}mg. \tag{20}$$

The “cosmological value” of $\alpha$ is $\alpha_\infty \approx 19$ [4]. However, $\alpha$ is a running constant, its value $0 \leq \alpha \leq \alpha_\infty$ dependent on distance and/or the mass of the gravitating object. The exact scaling law for $\alpha$ is not known at present, but we may observe that introducing a scaled value of $\alpha = \alpha(R)$ into Eq. (12) means, in effect, that a weighted average of $\alpha$ appears in subsequent equations, which we denote by $\bar{\alpha}$. As the scaling law for $\alpha$ is unknown, we are free to postulate that $\bar{\alpha} = 2$. Consequently, (20) now reads

$$F(g) \simeq -\frac{3}{2}\frac{\bar{\alpha}}{1+\bar{\alpha}}mg = -mg. \tag{21}$$

This is Newton’s law of inertia. The right-hand side of Eq. (21) represents the force $F = mg$ that acts on the test particle to accelerate it with $g$; as such, Eq. (21) is equivalent to d’Alembert’s principle [7], according to which the sum of all forces (inertial and noninertial) must be zero, i.e.,

$$F_1 + F = 0, \tag{22}$$

where we set the force of inertia, $F_1 = F(g)$.

The inertial force arose as a result of the influence of distant matter in the universe on the test particle, offering an effective realization of Mach’s principle [8].

We must emphasize that this law was recovered, in the weak field approximation of our modified gravity, from the gravitational potential alone; inertia was not postulated either explicitly or implicitly. We must further emphasize that in the final expression for $F(g)$, the present-day density of the universe or other cosmological parameters are no longer present. If the scaling law for $\alpha$ is itself scale-invariant, one can expect $\bar{\alpha}$ to remain constant as the universe evolves, and the law of inertia is not violated over time.

The mass $m$ on the right-hand side of Eq. (21) is the passive gravitational mass, characterizing how an object responds to an external gravitational field. We can also define the inertial mass of an object as

$$m_I = -\frac{F(g)}{g}. \tag{23}$$

The equivalence of the passive gravitational and inertial mass is often referred to as the weak equivalence principle.

For very small accelerations ($g \ll cH_0$), the approximation that led to Eq. (21) no longer applies. Instead, we can use \( \log (1 + x) = x - x^2/2 + x^3/3 + \mathcal{O}(x^4) \) (we need to expand to $x^3$, as the first two terms are canceled out in Eq. (17)), to get

$$F(g) \simeq -\frac{16\pi G \mu}{3\mu^2 c^2} \frac{\bar{\alpha}}{1+\bar{\alpha}}mg = -2\frac{\bar{\alpha}}{1+\bar{\alpha}}mg = -\frac{4}{3}mg. \tag{24}$$

This is also the force law we would have obtained from Eq. (15), for small accelerations, without applying a relativistic correction [6].

This is a profound result: we are, in fact, predicting a small violation of the weak equivalence principle for accelerations $g \sim cH_0$ (see Fig. 1) This is a testable prediction, especially in view of the fact that the acceleration $g$ in Eq. (17) does not need to be gravitational in origin.

Fig. 2 depicts schematically a simple experiment that could be used to verify the validity of Newton’s law of inertia at very small accelerations. The values of masses, lengths, time scales, and charges are not extreme in nature. However, we have to ensure that no non-gravitational forces act on the test particle other than the force being measured. Therefore, the experiment must be performed in a geodesic laboratory.

![FIG. 1: Does MOG violate the weak equivalence principle for very small accelerations? The horizontal axis in this plot is acceleration, measured in units of the “cosmic acceleration” $cH_0 \approx 7 \times 10^{-10}$ m/s$^2$. The vertical axis shows the difference between inertial mass $m_I = -F(g)/g$ and passive gravitational mass $m$, as predicted by Eq. (17).](image1)

![FIG. 2: Schematic of a simple experiment that can be used to verify the validity of the force law $F = mg$ for very small accelerations. With the values presented here, a measurement of a deflection of $\sim 1.8$ mm over the course of ten minutes with an accuracy better than $\sim 10\%$ is required, in order to measure the deficit in inertial mass. A smaller acceleration (corresponding with $E \approx 0.01$ V/m) could be used to measure an excess in inertial mass of up to $\sim 30\%$.](image2)
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