Lecture 3  September 13

- Review of HW #1 Answers, Difficulties & Expectations
- Review of last class (drifts)

- Last we saw: Example of drifts & Magnetic Mirrors

Today:

- Motion in a time varying magnetic field (Polarization Drift)
- Adiabatic invariants
- Example: Motion of Trapped Particles in the Magnetosphere
- Basic equations of MHD

\[ (E \times B) \]

Motions in static inhomogeneous magnetic fields: \( E \times B \) drifts, \( J \times B \) drifts, curvature drifts, changes in \( \mu \), due to mirror reflections.

Motion in a time varying magnetic field ([1] and pg 45-47 of [3])

If the magnetic field varies with time, \( J \times B = -\tau B \) and the total energy of the particle is not constant.

Assume that \( \tau B \) is varying on a time scale which is slow compared to \( \omega_c \).

A charged particle orbiting in this field gains energy per orbit:

\[ \Delta u_e = q \int \mathbf{E} \cdot d\mathbf{s} - q \int \frac{\partial \mathbf{B}}{\partial t} \cdot dA \]
Question of Puckt Bell -

where the current drift is pointing?

\( F_c = - \nabla \times B \) (refer page 35)

\[ V_0 = -K m_j v_i \times B^2 \]

\[ \text{correction from foil} \]

dependant on the charge!

You will find that for the Earth dipole -

\[ \frac{V_c}{V_{00}} = 2W_i \text{ at the equator} \]

\[ \frac{V_0}{V_{00}} \]
To a good approximation, the area is $A = \pi r^2$.

So,

$$dA = \frac{d}{dt} \left( \frac{1}{2} Br^2 \right) = \frac{1}{2} \frac{d}{dt} \left( Br^2 \right).$$

Since the cyclotron period is $\omega = \frac{2\pi}{v}$, the rate of change of the perpendicular energy is given by

$$\frac{dE}{dt} = \frac{1}{2} m v^2 \frac{d}{dt} \left( \frac{1}{2} Br^2 \right).$$

Also, since $E = \frac{\omega}{B}$

$$\frac{dE}{dt} = \frac{1}{2} m v^2 \frac{d}{dt} \left( \frac{1}{2} Br^2 \right) \implies \frac{dE}{dt} = \text{constant}.$$

Again, for a time dependent $B$, we found that the magnetic moment is a constant (same as for converging magnetic fields).

The conservation of the magnetic moment is equivalent to maintaining a constant magnetic flux through the cyclotron orbit.

$$\mu = \frac{1}{2} m v^2 = \frac{1}{2} \frac{q^2}{m} B$$

$$\Phi_B = \frac{1}{2} \pi r^2 B \text{ magnetic flux}$$

Since $\mu = e\ell$ $\Rightarrow$ $\Phi_B = e\ell$.
The drift is divided by the line and associated with acceleration in the area of the drift. It is associated with a change in the direction of motion, but not in average speed. It is considered to be the motion of the drift in the entire energy graph.
The motion under the influence of two combined fields gives zigzag along an oblique path. The particle moves from one potential to another gaining energy.
The change in the cyclotron orbit due to a time varying magnetic field (top), and a spatially varying magnetic field (bottom). In both cases the magnetic flux through the orbit is removed, i.e., $\Phi_{\text{in}} = \Phi_{\text{out}}$.


An adiabatic (very) slow change in the length of the pendulum energy and frequency is a constant of motion.

This is based on the notion that there are distinct spatial scales in the problem. One scale associated with the gyro-motion and another, much shorter scale, associated with field inhomogeneity.

Action variables $S = \oint P \cdot dQ$ (back in Classical Mechanics - Goldstein).

A generalized momentum conjugate to a periodic coordinate $Q$.

For non-relativistic particles in an electromagnetic field, the generalized momentum is $P = m \dot{Q} + \mathbf{q} \cdot \mathbf{A}$, when $\mathbf{B} = \nabla \times \mathbf{A}$.

$S$ is an invariant or constant of motion, provided changes in the variables occur slowly compared to the relevant periods of the system and the rate of change is almost zero.

The system change from one state of motion take another and still have the same $S$. 

Kalev - have a discussion on polarization and why plasma is good at shielding out electric fields. (Take this to the bank)
Consider the gyro motion in local cylindrical coordinates, \( r, \theta, z \) with \( B = B_0 e_z \), \( \mathbf{A} = \frac{1}{2} \mathbf{B} r e_\theta \) where the particle moves on an orbit \( r = R \cos \theta \) in the direction of decreasing \( \theta \).

Using Abakhan's Invariant, the Hamiltonian is

\[
H = \int \left( p_r^2 \right) r^2 \, dr = \frac{1}{2} m v_r^2 \frac{1}{r^2} \, dr = \frac{1}{2} m v_r^2 \frac{1}{R^2} \, \theta \, d\theta
\]

\[
= \frac{1}{2} m v_r^2 \frac{1}{R^2} \, \theta \, d\theta = \frac{1}{2} m v_r^2 \frac{1}{R^2} \, \theta \, d\theta
\]

\[
J = \int \frac{1}{2} m v_r^2 \frac{1}{R^2} \, \theta \, d\theta = \frac{1}{2} m v_r^2 \frac{1}{R^2} \, \theta \, d\theta
\]

\[
= \frac{1}{2} m v_r^2 \frac{1}{R^2} \, \theta \, d\theta
\]

The assumption that \( J \) is constant is that the external parameter only vary slowly.

Example: Magnetic Mirror

Because \( p_r \) cannot, if the particle moves into the mirror \( V_r \) must increase, since \( B \) increases.

Because of energy conservation, this can happen only if \( V_r \) decreases.
Reflection of particles in the magnetic mirror.

We saw that particles with

$$\theta < \theta_0 \implies r_{cm} = \frac{m}{B}$$

are lost to a loss cone.

Particles outside the loss cone are trapped. They bounce back and forth between the mirrors.

Second Adiabatic Invariant: Longitudinal Invariant

$$\sum m v_z d z = \oint p_z d z = \oint m v_z^2 d z = \oint m v_z^2 dt = \text{const.}$$

where the motion along the z axis has been assumed to be a harmonic oscillator with

$$v_z = \sqrt{2} \alpha \cos \omega t$$

$$= \sqrt{\frac{2 \alpha}{m}}$$

First Term: mechanism. The reflection of a particle at a fixed magnetic mirror is equivalent to a ball bouncing at a wall. In a magnetic bottle with 2 fixed mirrors, the particle oscillates between these mirrors without changes in total energy. The interaction with a moving magnetic mirror is equivalent to the reflection off a moving wall.

Depending on the relative speeds between particle and mirror, an energy gain or loss results: head-on collisions lead to energy gain

If the particle and mirror move in the same direction, \( \rightarrow \) energy loss.
From \( S_2 = \int \frac{m v_{//} \text{dl}}{S_0} \)

In the total energy gain is determined by the shortening of the distance between the mirrors. (Similarly, equivalent: warming of a gas during compression, etc., pumps).

Let's consider cases where the magnetic field is not axially symmetric.

Since gradient of a radially drift gives an azimuthal drift motion which field have a particle will be after it drifted to a new azimuthal location?

If \( B = B_0 \) in time then from energy conservation \( B_m \) ce k \( \frac{2}{4} m v^{2} = e B(\frac{1}{2}) \)

\( S_2 = \int \frac{\text{d}l}{B_m} \left( \frac{1 - B(\frac{1}{2})}{B_m} \right) = 4 \pi L_0 \)

As the particle drifts in azimuth, the locations of the new mirrors and determined by:

1) \( B_m = e k = B_{\text{init}} \)
2) \( S_0 = e L_0 = S_{\text{init}} \)

The law of all field lines around the planet with a name for a gun, \( B_m \) generates a surface called \( L = L_0 \).

**Figure 3-70**: As a particle drifts in azimuth in a planetary magnet.
Third Adiabatic Invariant: The Invariant

Third adiabatic invariant is associated with azimuthal drift motions and only exists for fields with well-defined axial symmetry, so the particles drift around in nearly closed orbits.

Drifts are caused by curl of the gradient of B.

The third invariant states that the magnetic flux enclosed by the surface in which the drift is constant is constant.

In Summary:

\[
\begin{align*}
\mathbf{J}: & \text{gyro} \\
\mathbf{E:} & \text{bouncing} \\
\mathbf{B:} & \text{curl} \\
J_\phi: & \text{invariant}
\end{align*}
\]

(a) $e$ and ions execute a fast gyration in opposite directions about the B axis conserving $J_\phi$ (magnetic moment).
(b) They bounce back and forth between mirror lines on a space-time scale, conserving $J_\phi$.
(c) They drift on a slower time scale yet in opposite longitudinal directions conserve the 3rd adiabatic invariant $J_\phi$ (the magnetic flux inside the drift shell).

Note: the barycentric rotation of the solar wind with the magnetosphere will not increase $J_\phi$, since its fast but easily will invalidate the variance of $J_\phi$.

Some numbers for 10 keV proton placed at $r = 5 R_e$

- \( B(\text{equator}) = 0.3 \text{ G at } R_e \)
- \( B(5 R_e) = 0.004 \text{ G at } 5 R_e \)
- \( n = 2 \times 10^4 \text{ cm}^{-3} \text{ at } 5 R_e \)

\( v = 24 \text{ km/s} \text{ at } 5 R_e \)
\[ v = 4.6 \times 10^8 \text{ cm/sec} \rightarrow R = 6.7 \text{ km (aprameter)} \]

If \( \theta = 90^\circ \) it will haul \( R_e = 6 \times 10^8 \text{ cm} \) along the line so it forms line \( a \)

\[ \frac{\text{v}}{\text{r}_e} \approx 100 \text{ sec} \]

\[ V_0 = \frac{3p}{5} \approx 6 \text{ km} \]

\[ \frac{3p}{5} \]

so it takes \( \frac{a}{v} \approx 5 \times 10^7 \text{ sec} \) to circle the earth \( \approx 5 \text{ km/sec} \)

This for a particle that are three times:

- cyclotron time \( \approx 0.2 \text{ sec} \)
- bound time \( \approx 100 \text{ sec} \)
- drift time \( \approx 3.6 \times 10^7 \text{ sec} \)

V. Example of violation of invariants:

- Parameters of the system varying too fast
- System is not longer periodic (action integral no longer exist).

Violation of 1st invariant

- Abrupt change in \( B \) such as in shock waves & other discontinuities

Violation of 2nd invariant

![Diagram showing violation of the 2nd adiabatic invariant due to escape of the particle during a parallel compression.]

The particle cannot increase its point that it can no longer remain trapped.

Mechanism
Violations of the 3rd invariant

Any change in the system that occurs on a time scale comparable to short
than the azimuthal drift period violates the 3rd invariant.

Planetary magnetospheres: changes in electric and magnetic fields due to
variations in solar wind pressure.

If the 3rd adiabatic invariant is violated, particles can move from one
L-shell to another as they drift azimuthally around the planet.

![Diagram of magnetic field lines and particles](image)

**Figure 3.34** Violations of the 3rd adiabatic invariant can cause particles to drift
anomally into different L-shells (L1 to L2). If the 3rd adiabatic invariant is con-
considered, the increase in the magnetic field strength (B1 to B2) causes a considerable
increase in the perpendicular energy, \( \frac{eB^2}{2m} \).

If a particle diffuse into a dipole magnetic field from a large distance
while conserving its 1st (\( L = \frac{m\omega}{eB} \)) and 2nd (\( S = m\sqrt{K} \)) adiabatic
invariants—considerable energization can occur:

\[
B(dipole) \propto \frac{1}{R^3} \quad \text{so} \quad \frac{W_1}{R^3} = \text{constant} \]

As a particle moves from \( L_1 \) to \( L_2 \),

\[
E + \langle \frac{W_1}{L_2} \rangle
\]

Since the length of the magnetic field decreases in going from \( L_1 \) to \( L_2 \),

\[ V_{L2} \] will increase as well.
MAGNETOTOROIDYDYNAMICS (Chapter 4)

If the plasma is sufficiently collisional \( \rightarrow \) MHD

- Mass density \( \rho \)
- Momentum density \( \rho \mathbf{v} \)
- Pressure \( P \)
- Electric field \( \mathbf{E} \)
- Magnetic field \( \mathbf{B} \)

All functions of \( r, \theta, \phi \)

(How to go from distribution function to the MHD equations we will see later)

\[ n(r, \theta, \phi) = \int \frac{k(v)}{\mathbf{v}} \, dv \]

Basic equations of MHD

Mass density is the sum of the mass densities of the individual species

\[ \rho_m = \sum \rho_m \quad \text{where} \quad \rho_m = m_m n_m \quad \text{(1)} \]

Fluid velocity

\[ \mathbf{v} = \sum \rho_m \mathbf{v}_m \quad \text{mass weighted average of the} \quad \text{(2)} \]

Note: since the ions have a much greater mass than the electrons, the fluid velocity \( \mathbf{v} \) is mostly weighted by the velocity of ions.

Mass Continuity Equation

\[ \frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{v}_m) = 0 \]
where \( \rho_s = m_s/V_s \)

Summing over all species \( s \) and using Eqs. (I) and Eq. (II) we obtain:

\[
\frac{\partial \rho_s}{\partial t} + \nabla \cdot (\rho_s \mathbf{V}_s) = 0
\]

where \( \mathbf{V}_s \) is the mass-weighted average of the fluid velocities of the individual species.

Momentum Equation:

\[
\frac{\partial}{\partial t} (m_s V_s) + \nabla \cdot (m_s V_s \mathbf{V}_s) = n_s c_s \left( T + V_s \times B \right) - V_s \cdot \mathbf{B} + \epsilon_s \frac{\partial \mathbf{E}}{\partial t}
\]

Summing over all species:

\[
\frac{\partial}{\partial t} \left( \sum_s m_s V_s \right) + \nabla \cdot \left( \sum_s m_s V_s \mathbf{V}_s \right) = \rho_s \frac{\partial \mathbf{E}}{\partial t} + \sum_s \epsilon_s \frac{\partial \mathbf{E}}{\partial t}
\]

where \( \rho_s = \sum_s m_s c_s \) and \( \sum_s \epsilon_s \)

Since the total momentum is conserved for any collision process:

\[
\sum_s \frac{\partial \mathbf{E}}{\partial t} = 0
\]

On the pressure tensor:

\[
\mathbf{P}_s = \rho_s V_s \mathbf{W}_s \mathbf{W}_s
\]

\( \mathbf{W}_s = \mathbf{W}_s - \mathbf{U}_s \) average velocity of the \( s \)th specie relative to the fluid velocity

\( \mathbf{U}_s \) motion at the fluid velocity

\( \rho_s \) pressure in a frame moving at

The total pressure \( \mathbf{P}_s = \sum_s \mathbf{P}_s \)
we get \[ \frac{\partial}{\partial t} \rho (\mathbf{u}, \mathbf{w}) + \frac{1}{3} \nabla \cdot (\rho \mathbf{u} \mathbf{w}) = \mathbf{F}^s \cdot \mathbf{w} - \mathbf{P}_o \] (iv)

With the subduction \( \mathbf{w} = \mathbf{w}_0 + \mathbf{v} \), the LHS of (iv) can be written as \[ \frac{\partial}{\partial t} \rho (\mathbf{u}, \mathbf{w}) + \frac{1}{3} \nabla \cdot (\rho \mathbf{u} \mathbf{w}) + \nabla \cdot (\rho \mathbf{u} \mathbf{w}_0) + \mathbf{P}_o \cdot \nabla \cdot (\rho \mathbf{u} \mathbf{w}) \]

Summation over all species and using \( \sum \rho \mathbf{w}_0 = 0 \) the LHS becomes \[ \frac{\partial}{\partial t} \rho (\mathbf{u}, \mathbf{w}) + \frac{1}{3} \nabla \cdot (\rho \mathbf{u} \mathbf{w}) + \frac{1}{3} \nabla \cdot (\rho \mathbf{u} \mathbf{w}_0) \]

Substituting that in (iv) \[ \frac{\partial}{\partial t} \rho \mathbf{u} + \nabla \cdot (\rho \mathbf{u} \mathbf{w}) = \rho \mathbf{F}^s + \frac{1}{3} \nabla \cdot \mathbf{B} - \mathbf{P}_o \]

Using the continuity equation \( \rho \mathbf{u} d\mathbf{v} \)

\[ \rho \mathbf{u} \left[ \frac{\partial}{\partial t} (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla \cdot (\mathbf{u} \cdot \nabla) \mathbf{u} \right] = \rho \mathbf{F}^s + \frac{1}{3} \nabla \cdot \mathbf{B} - \mathbf{P}_o \]

The rate of change of the momentum of a fluid element is equal to the sum of the electric field, magnetic field, and pressure forces acting on that fluid element.

\( \mathbf{P}_o \rightarrow \mathbf{P}_o \) (Pressure is always defined relative to the fluid frame of reference)
Generalized Ohm's Law

How the current is related to the electric field $E$.

A complete rigorous derivation is not possible. In an attempt, let us consider a two-component plasma: electrons, and one positive charged ion ($i$).

Let assume $\rightarrow$ collisions $\rightarrow$

$$\frac{\partial \mathbf{E}}{\partial t} = -n_{i} m_{i} v_{i} \mathbf{E}(\mathbf{u}_{i} - \mathbf{u}_{e})$$

Electric collision frequency

between the two species

This is valid only if the velocity difference $\mathbf{u}_{i} - \mathbf{u}_{e}$ is small compared to the thermal velocities of the two species.

The momentum equation for the ions is given by

$$\frac{\partial}{\partial t} \left( m_{i} n_{i} \mathbf{u}_{i} \right) + \nabla \cdot \left( m_{i} n_{i} \mathbf{u}_{i} \mathbf{u}_{i} \right) = -n_{i} e \left[ \mathbf{E} + \mathbf{u}_{i} \times \mathbf{B} \right] +$$

$$- \mathbf{B} \cdot \mathbf{J} - m_{i} n_{i} \mathbf{v}_{i} \left( \mathbf{u}_{i} - \mathbf{u}_{e} \right)$$

and the momentum equation for the electrons:

$$\frac{\partial}{\partial t} \left( m_{e} n_{e} \mathbf{u}_{e} \right) + \nabla \cdot \left( m_{e} n_{e} \mathbf{u}_{e} \mathbf{u}_{e} \right) = -n_{e} e \left[ \mathbf{E} + \mathbf{u}_{e} \times \mathbf{B} \right] +$$

$$- \mathbf{B} \cdot \mathbf{J} - m_{e} n_{e} \mathbf{v}_{e} \left( \mathbf{u}_{e} - \mathbf{u}_{i} \right)$$

Multiplying eq. (v) by $m_{e}$, eq. (vi) by $m_{i}$ and subtracting:

$$\frac{\partial}{\partial t} \left[ m_{i} n_{i} \left( \mathbf{u}_{i} - \mathbf{u}_{e} \right) \right] + \nabla \cdot \left[ m_{i} n_{i} \left( \mathbf{u}_{i} \mathbf{u}_{i} - \mathbf{u}_{e} \mathbf{u}_{e} \right) \right]$$

$$= \rho_{m} \mathbf{E} + e \gamma \left( m_{i} n_{i} \mathbf{u}_{i} + m_{e} n_{e} \mathbf{u}_{e} \right) \times \mathbf{B} - m_{e} \mathbf{B} \cdot \mathbf{J} + m_{i} \mathbf{B} \cdot \mathbf{J}$$

$$- m_{i} n_{i} \mathbf{v}_{i} \left( \mathbf{u}_{i} - \mathbf{u}_{e} \right) +$$

where $\rho_{m} = n \left( m_{i} + m_{e} \right)$

$\rho_{m} n_{i} \mathbf{v}_{i} \left( \mathbf{u}_{e} - \mathbf{u}_{i} \right)$

(VII)
Since the collisions "drag force" on the electrons should be equal and
opposite on the ions,
\[ m, v_a = m, v_e \]
the last two terms of (\ref{eq:7}) become

\[ -e \mu \frac{\partial}{\partial t} (v_i - \bar{v}_e) + m, v_a \frac{\partial}{\partial t} (\bar{u}_e - \bar{u}_i) = e \mu (m, v_i - \bar{v}_e) \]

where \( \bar{z} = \frac{\mu}{n} (\bar{u}_i - \bar{u}_e) \)
The result is
\[ \bar{z} = \frac{\mu}{e} \frac{m, v_e}{m, v_i} \]

6. Conductivity

(\ref{eq:7}) dissipated by collisions - energy is being dissipated
The energy dissipation occurs because the ordered motion
produced by the applied electric field is converted into random thermal
motion (i.e., heat) by collisions.

The LHS of (\ref{eq:7}) the term \( m, \bar{u}_i + m, \bar{u}_e \) can be written as

\[ m, \bar{u}_i - m, \bar{u}_e = \frac{m, v_e}{m + m, v_i} \bar{u}_i - \frac{m, v_e}{m + m, v_i} \bar{u}_e \]

with \( \bar{x} = \frac{m, \bar{u}_e}{m + m, v_i} \bar{u}_e = \frac{m, v_e}{m + m, v_i} \bar{u}_e \)

The LHS of (\ref{eq:7})
\[ \frac{\partial}{\partial t} \left[ \left( \bar{v}_i - \bar{u}_e \right) \right] = \frac{\mu}{e} \frac{m, \bar{v}_i}{\bar{z}} \]

\[ \bar{z} = \frac{\mu}{n} (\bar{u}_i - \bar{u}_e) \]
Now, the term \( V \cdot [\dot{v}(\mathbf{U}_e - \mathbf{U}_e)] \)

... but remember that we assumed that \( \delta \mathbf{U} = \mathbf{U}_e - \mathbf{U}_i \) is small so we can expand

\( \mathbf{U}_e - \mathbf{U}_i \) to second order in terms of \( \delta \mathbf{U} \)

to give in first approximation

\[
\mathbf{V} \cdot \nabla (\mathbf{U}_e - \mathbf{U}_i) = \frac{1}{6} \mathbf{V} \cdot (\delta^2 + \mathbf{U}_e^2)
\]

where we set \( \mathbf{U}_i = \mathbf{U}_e \) for \( \mathbf{U}_i \) small. 

Substituting all that \( \mathbf{V} \cdot \mathbf{U}_e \) and ignoring \( \mathbf{P}_e \) since it is reduced by a factor of \( \rho \) in relationship to \( \rho \mathbf{U} \) we obtain

\[
\frac{\mathbf{V} \cdot \dot{\mathbf{U}} + \mathbf{U} \times \mathbf{E}}{\delta} = \frac{1}{6} \left( \frac{3}{\rho} \mathbf{E} \cdot \mathbf{E} + \frac{m_e}{\rho^2} \frac{J^2}{\rho^2} \right)
\]

Generalized Ohm’s law

If all the terms on the RHS are very small, the equation is the simple form of Ohm’s law

\( \mathbf{J} = \sigma (\mathbf{E} + \mathbf{U} \times \mathbf{B}) \)

Away from “boundary layers” where \( \mathbf{E} \) is large, the terms on the right hand side are small and usually are neglected (we will return to this later).

As the collision frequency to zero \( \mathbf{E} \rightarrow \infty \) and Ohm’s law reduces to

\( \mathbf{E} + \mathbf{U} \times \mathbf{B} = 0 \)

A plasma that obeys this equation is called an ideal plasma.
So the fluid velocity component perpendicular to $\mathbf{B}$ is then

$$\mathbf{u} = \frac{\mathbf{E} \times \mathbf{B}}{B^2}$$

which is identical to $E \times B$ drift velocity.

Also, for an ideal MHD plasma,

$$\mathbf{E} = 0$$

In collisional plasmas, it is assumed that all the terms in the right of the generalized Ohm’s law are negligible. This leads to the resistive form of Ohm’s law.

$$\mathbf{E} + \mathbf{U} \times \mathbf{B} = \frac{\mathbf{j}}{\sigma}$$

This equation is the simplest derivation from the ideal MHD model.

The Equation of State

The moment equations are not a closed system of equations so we must choose an equation of state to close it. The equation of state specifies

$$\rho \rightarrow \Gamma \rho$$

If there are sufficient collisions to establish an isothermal Maxwellian velocity distribution, the plasma pressure is isothermal

$$\mathbf{j} = \mathbf{v} P$$

Under these conditions, the equation of state is assumed to be

$$\frac{\partial (\rho \mathbf{v}^3)}{\partial t} = 0$$

$\Gamma = \text{polytropic index}$
\[ \frac{\partial}{\partial t} \left( \rho \mathbf{v} \right) = 0 \rightarrow \text{adiabatic equation of state} \]

Other eq. of state:
- Isothermal: \( \gamma = 1 \)
- Incompressible: \( \gamma = \infty \)

If there are insufficient collisions to maintain an isothermal velocity distribution, \( P \) is anisotropic \( \mathbf{P} \) (tensor)

If the particle motions in the rest frame of the fluid are axially symmetric with respect to the magnetic field, the pressure tensor can be represented as a matrix:

\[
\mathbf{P} = \begin{bmatrix}
P_z & 0 & 0 \\
0 & P_z & 0 \\
0 & 0 & P_z
\end{bmatrix}
\]

where \( P_z \): pressure perpendicular to \( \mathbf{B} \)

\( P_\parallel \): pressure parallel to \( \mathbf{B} \)
Field Equation + Ohm's law + Maxwell's Equations  
(assuming $\rho_i = 0$) 

assuming that the pressure is scalar and adopt the eq. of state: 

\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J} \quad \text{(Ampère's law)} \]
\[ \nabla \cdot \mathbf{B} = 0 \]
\[ \nabla \times \mathbf{E} = -\partial \mathbf{B}/\partial t \quad \text{(Faraday's law)} \]
\[ \nabla \cdot \mathbf{E} = 0 \quad \text{(Gauss's law)} \]
\[ \frac{\partial \rho_m}{\partial t} + \nabla \cdot (\rho_m \mathbf{U}) = 0 \quad \text{(mass continuity equation)} \]
\[ \rho_m \frac{\partial \mathbf{U}}{\partial t} = \nabla \times \mathbf{B} - \nabla P \quad \text{(momentum equation)} \]
\[ \mathbf{S} = \sigma (\mathbf{E} + \mathbf{U} \times \mathbf{B}) \quad \text{(Ohm's law)} \]

and
\[ \frac{\partial (\rho_m \mathbf{U})}{\partial t} = 0 \quad \text{(equation of state)} \]