Relativistic dynamics

- Lorentz transformations also affect the accelerated motion of objects under the influence of forces. In Newtonian physics a constant force F accelerates an abject at a constant rate $a = dv/dt = F/m$, so that the velocity $v = at$ of the object can become arbitrarily large if one waits long enough. This cannot be in the relativistic physics because nothing can move faster than light. The acceleration must become smaller and smaller as the velocity approaches c, which can be interpreted as the growth of mass m. In this interpretation the velocity-dependent mass $m(v)$ is sometimes called relativistic mass, while the mass at rest $m = m(0)$ is called rest mass.

- We will derive the expression for $m(v)$ mostly from kinematic principles. Consider two identical objects of (rest) mass m. Let us label by B the reference frame fixed to one of the objects (#1) so that the other object (#2) moves away from it along x axis with velocity v. The trajectory of the moving object is $x_B(t_B) = vt_B$. The center of mass of these two objects is located at:

$$X_B = \frac{m(0)\times 0 + m(v)\times x_B}{m(0)+m(v)} = \frac{m(v)}{m+m(v)}vt_B$$

  and moves with velocity

$$u = \frac{dX_B}{dt_B} = \frac{m(v)}{m+m(v)}v$$

- Now set up a reference frame A fixed to the center of mass of the two objects. This frame moves with velocity u relative to B along x axis, producing the same set up for Lorentz transformations as before. The object #1 which is at rest in B simply moves with velocity $-u$ in A. The other object #2 must then move with velocity $+u$ in the frame A. This follows from the requirement that the center of mass appear at rest in the frame A, so that the two objects must move in A with the same velocity u in opposite directions (and their relativistic masses $m(u)$ will be the same).

- We can relate the velocities of the object #2 in the frames A and B by using the velocity transformation rule ($v_A = u$, $v_B = v$):

$$v_B = \frac{v_A + u}{1 + v_A u/c^2} \qquad \Longrightarrow \qquad v = \frac{2u}{1+u^2/c^2}$$

  Let us write $v = \alpha u$ and obtain the equation for α:

$$1 = \frac{v}{2u}\left(1+\frac{u^2}{c^2}\right) = \frac{v}{2u}\left(1+\frac{u^2}{v^2}\frac{v^2}{c^2}\right) = \frac{\alpha}{2}\left(1+\frac{1}{\alpha^2}\frac{v^2}{c^2}\right)$$

$$1+\frac{v^2}{c^2}\frac{1}{\alpha^2} = \frac{2}{\alpha}$$

$$\alpha^2 - 2\alpha + \frac{v^2}{c^2} = 0$$

  This quadratic equation has solutions:

$$\alpha = 1 \pm \sqrt{1-\frac{v^2}{c^2}}$$

  We cannot allow the center of mass to move faster than the object #2 in the frame B, so we must take the solution with the plus sign ($\alpha > 1 \Leftrightarrow v > u$).

- Finally, we go back to the expression which relates u, v and the masses:

$$\alpha = \frac{v}{u} = \frac{m+m(v)}{m(v)} = 1 + \frac{m}{m(v)} = 1 + \sqrt{1-\frac{v^2}{c^2}}$$

  We find that

$$m(v) = \frac{m}{\sqrt{1-\frac{v^2}{c^2}}}$$

- What role exactly does $m(v)$ play in dynamics? The only physical purpose of mass is to establish a relationship between dynamics (forces) and kinematics (motion) of bodies, and our real goal here is to discover how relativistic objects respond to forces.. In order to answer this question we must recognize that dynamics implicitly enters the discussion above through the center-of-mass. For two bodies with momenta p_1 and p_2 (in some reference frame) and masses m_1 and m_2 respectively, the center of mass is a fictitious point which carries the total momentum $P = p_1 + p_2$ the way a particle with the total mass $M = m_1 + m_2$ would, that is with velocity $u = P/M$. Unless there are external forces, the total momentum is conserved:

$$\frac{dP}{dt} = \frac{dp_1}{dt} + \frac{dp_2}{dt} = F_{2\to1} + F_{1\to2} = 0$$

because the forces $F_{i\to j}$ exerted by the object i on the object j cancel out according to the third Newton's law (the principle of action and reaction). We now want to define mass by expressing a particle momentum as the product of its mass and velocity, $p_i = m_i v_i$. The total momentum is

$$P = m_1 v_1 + m_2 v_2 = Mu$$

and its conservation implies $u =$ const. no matter how the two particles interact with each other. This $u =$ const. is what allowed us to apply Lorentz transformations to the (inertial) center-of-mass frame. Therefore, the above calculation is implicitly consistent with the second Newton's law

$$F = \frac{dp}{dt}$$

and the definition of mass from $p = m(v)v$.

- To summarize, the momentum of a relativistic object moving at velocity v is:

$$p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$$

where m is the object's "rest" mass (in modern terminology, m is simply called mass). If the object is slow, we can neglect v/c so its momentum is simply $p = mv$. On the other hand, by applying the constant force F we can build up arbitrarily large momentum $p = Ft$ in sufficiently large time interval t. As $p \to \infty$, the velocity v approaches the speed of light but never exceeds it.

Relativistic energy

- Here we seek the relationship between energy and momentum in the special theory of relativity.

- Consider a body of ("rest") mass m initially at rest. Then, let us turn on at time $t = 0$ a constant force F which accelerates the body. From the second Newton's law we immediately obtain:

$$\frac{dp}{dt} = F \qquad \Rightarrow \qquad p = Ft$$

since the force is constant, and $p = 0$ at $t = 0$. The kinetic energy T acquired by the time t is equal to the work Fx done by the force F, where $x(t)$ is the position of the body. Since the velocity is $v = dx/dt$, we can obtain position by integrating out velocity:

$$T = Fx = F \int_0^t dt' \, v(t')$$

- Let us express the current velocity v in terms of momentum p. We can take the square of

$$p = \frac{mv}{\sqrt{1 - \frac{v^2}{c^2}}}$$

and obtain

$$m^2v^2 = p^2\left(1-\frac{v^2}{c^2}\right)$$

$$v = \frac{p}{\sqrt{m^2+\frac{p^2}{c^2}}} = \frac{pc}{\sqrt{m^2c^2+p^2}}$$

- Substitute this in the expression for energy:

$$T = F\int\limits_0^t dt'\,\frac{p(t')c}{\sqrt{m^2c^2+p(t')^2}}$$

and note that we can pull the constant force inside the integral and combine it with time, $Ft' = p(t') \equiv p'$. At the desired final time t the momentum p' becomes p. The above integral can be therefore written as:

$$T = c\int\limits_0^p dp'\frac{p'}{\sqrt{m^2c^2+p'^2}}$$

This is easy to solve. We first change the integration variable $\eta = p'^2$ by noting that $p'dp' = d\eta/2$. Then we use the integration formula:

$$\int d\eta(a+\eta)^n = \frac{(a+\eta)^{n+1}}{n+1}$$

for $n = -1/2$ and $a = m^2c^2$. The result is:

$$T = \frac{c}{2}\int\frac{d\eta}{\sqrt{m^2c^2+\eta}} = \frac{c}{2}\times 2\sqrt{m^2c^2+p'^2}\Big|_{p'=0}^{p'=p} = \sqrt{(mc^2)^2+(pc)^2}-mc^2$$

- Even though this expression was derived by considering constant acceleration, it holds regardless of how the force F changes in time. The Newtonian principle of converting work to energy is merely a formal device to define energy as a conserved quantity. We could have turned off the force F for a while and continued accelerating later; both momentum and energy would not change during the wait time. Therefore, energy can depend only on momentum (and mass) of the body.

- It is customary to define the total energy of a body as

$$E = \sqrt{(pc)^2+(mc^2)^2}$$

where

$$E_0 = mc^2$$

is considered to be the energy at rest when momentum $p = 0$. This is the famous Einstein's equation which hints that the mere mass of objects could be a source of energy. In Newtonian physics mass is conserved, so one could not extract energy from mass and use it to do some work. However, the theory of relativity already allowed us to view mass as something that can exchange energy with environment: when we accelerate an object whose velocity approaches the speed of light, the kinetic energy must be dumped into the increasing "relativistic mass" because the speed of light cannot be exceeded. The true power of this conversion between mass and energy is unleashed in nuclear reactions.

- How do we obtain the non-relativistic limit? At small velocities v momentum p is small so that $pc \ll mc^2$. Expanding the square root as $\sqrt{1+x} \approx 1+x/2$ for small x we obtain:

$$T = \sqrt{(mc^2)^2+(pc)^2}-mc^2 = mc^2\sqrt{1+\left(\frac{pc}{mc^2}\right)^2}-mc^2 \approx \frac{p^2c^2}{2mc^2} = \frac{p^2}{2m}$$

which is the familiar expression for kinetic energy in Newtonian physics.

Covariant formulation of the special theory of relativity

- The Einstein's first postulate requires that all laws of physics be the same in all inertial frames of reference. We say that the laws of physics must be invariant under Lorentz transformations. In this section we discuss a powerful formalism which allows one to formulate the laws of nature in a manifestly covariant way, so that they automatically keep their form under Lorentz transformations.

- The foundation of the covariant formulation is the so called Minkowski space. This is a four-dimensional combination of space and time with appropriately defined metric for expressing distances between points. The coordinates in Minkowski space are 4-vectors $(ct, x, y, z) \equiv (ct, \mathbf{r})$ which we label by x_μ. The subscript μ indexes any of the four coordinates.

- The metric tensor is defined by a matrix

$$g_{\mu\nu} = g^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

and used in the following fashion. The length of a vector x_μ in Minkowski space is defined by $L(x) = \sqrt{\sum_\mu x_\mu x^\mu}$ where x^μ is defined as

$$x^\mu \equiv \sum_{\mu\nu} g^{\mu\nu} x_\nu$$

In general, vectors can be specified with a lower or upper index, and $g_{\mu\nu}$ or $g^{\mu\nu}$ is used to convert between lower and upper index vectors by the convention that a lower index can be contracted only with an upper index in a summation over indices.

- If there weren't for the -1 element in $g_{\mu\nu}$, the vector length would be the same as in any Euclidean vector space (like our three-dimensional space). However, the given form of $g_{\mu\nu}$ is needed in order to make a vector length invariant under Lorentz transformations:

$$L^2 = \sum_{\mu\nu} x_\mu x_\nu g^{\mu\nu} = x^2 + y^2 + z^2 - (ct)^2$$

Under Lorentz transformations

$$\begin{aligned} ct' &= \gamma(\beta x + ct) \\ x' &= \gamma(x + \beta ct) \\ y' &= y \\ z' &= z \end{aligned}$$

the length becomes:

$$\begin{aligned} L' &= x'^2 + y'^2 + z'^2 - (ct')^2 \\ &= \gamma^2(x^2 + \beta^2 c^2 t^2 + 2\beta xct) + y^2 + z^2 - \gamma^2(\beta^2 x^2 + c^2 t^2 + 2\beta xct) \\ &= \gamma^2(1-\beta^2)x^2 + y^2 + z^2 - \gamma^2(1-\beta^2)(ct)^2 \end{aligned}$$

and since $\gamma = 1/\sqrt{1-\beta^2}$ we find that $L' = L$.

- Since 4-vector length is invariant under Lorentz transformations, we can interpret Lorentz transformations as four-dimensional rotations in Minkowski space. In tensor notation the Lorentz transformations are:

$$x'_\mu = \sum_\nu \alpha_\mu^\nu x_\nu$$

where

$$\alpha_\mu^\nu = \begin{pmatrix} \gamma & \beta\gamma & 0 & 0 \\ \beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

Any scalar product $\sum_\mu a_\mu b^\mu$ of two 4-vectors is automatically invariant under Lorentz transformations.

- An interval s_{12} between two events described by Minkowski vectors $x_{1\mu}$ and $x_{2\mu}$ is defined by

$$s_{12}^2 = \sum_\mu (x_{1\mu} - x_{2\mu})(x_1^\mu - x_2^\mu)$$

If the two events occur at the same location, then the interval is $s^2 = -c^2(t_1 - t_2)^2 < 0$. If they occur at the same time, the interval is $s^2 = (\mathbf{r}_1 - \mathbf{r}_2)^2 > 0$. An interval is called time-like if $s^2 < 0$ and space-like if $s^2 > 0$.

- Two events separated by a space-like interval cannot be causally connected. Clearly, one can find a reference frame in which such events occur simultaneously at different locations so that physical influence has no chance to propagate from one location to the other. Since the laws of physics must be the same in all reference frames, they must agree in all frames that the two events cannot be a cause or consequence of each other.

- We define proper time τ as the time measured in the reference frame fixed to an object whose motion we want to track. We can express the trajectory of the object in some reference frame $x_\mu(t)$. However, if we attempt to define velocity as dx_μ/dt we find that it does not transform between reference frames according to Lorentz transformations. This is so because x_μ transforms properly, but t transforms as well. On the other hand,

$$v_\mu = \frac{dx_\mu}{d\tau}$$

is a valid 4-vector in Minkowski space which transforms according to Lorentz transformations because the infinitesimal intervals $d\tau$ are Lorentz-invariant. This is the covariant definition of velocity. In a similar fashion we define covariant acceleration a_μ, momentum p_μ and force F_μ (m is the "rest" Lorentz-invariant mass):

$$a_\mu = \frac{dv_\mu}{d\tau} \quad , \quad p_\mu = mv_\mu \quad , \quad F_\mu = \frac{dp_\mu}{d\tau} = ma_\mu$$

These equations specify the complete relativistic dynamics in a form which is automatically compatible with Lorentz transformations.

- In terms of familiar quantities from Newtonian physics, three-dimensional position $\mathbf{r}$, velocity $\mathbf{v}$, acceleration $\mathbf{a}$, and momentum $\mathbf{p}$ vectors, the covariant 4-vectors are:

$$\begin{aligned} x_\mu &= (ct, \mathbf{r}) \\ v_\mu &= (\gamma c, \gamma \mathbf{v}) \\ a_\mu &= \left(\gamma \frac{\mathbf{F}\mathbf{v}}{mc}, \gamma \mathbf{a}\right) \\ p_\mu &= \left(\frac{E}{c}, \mathbf{p}\right) \\ F_\mu &= \left(\gamma \frac{\mathbf{F}\mathbf{v}}{c}, \gamma \mathbf{F}\right) \end{aligned}$$

with

$$\gamma = \frac{1}{\sqrt{1 - v^2/c^2}}$$

Einstein's general theory of relativity

- The special theory of relativity is a complete description of dynamics in inertial reference frames. However, already at the level of this definition there is a fundamental problem. An inertial reference frame is defined as a frame in which all forces acting on objects originate from other objects in a manner that depends on the objects' positions, velocities, etc. How can we tell for sure that the net force on a body has such a proper origin? All we can do is measure the force based on the body's acceleration. If we knew all laws of nature exactly and could measure the positions and velocities of all bodies in the universe, perhaps we could test whether the measured net force is a combination of forces with proper origins. Another way to appreciate the problem is to note that a frame moving uniformly with respect to an inertial frame is itself inertial. In order to ultimately establish that our frame of interest is inertial, we must be able to find its velocity relative to a universal special frame which is inertial for sure. The special theory of relativity denies the existence of such a preferred frame.

- Other problems can be found in examples containing accelerated motion. For instance, consider a planet orbiting a very heavy compact star at a relativistically large velocity. What is the radius of the orbit perceived from the planet's surface? The distance to the star, radius r, is always measured in the direction perpendicular to the planet's motion. However, the orbit perimeter appears shorter than $2\pi r$ by a factor $\sqrt{1-v^2/c^2}$ to the observer on the planet, even though the orbit looks circular to him. On the other hand, an observer hovering at one place on the surface of the star can clearly see that the planet moves at the distance r along the perimeter of length $2\pi r$. Whose judgment of space geometry is wrong? Surely, we cannot trust the predictions of special relativity for the observer on the planet, because his reference frame is not inertial (rotation is accelerated motion).

- The famous twin paradox illustrates a similar problem. One twin brother stays on Earth while the other one takes a space round-trip from Earth to a nearby star and back at near light-speed velocities. The guy staying on earth watches broadcast video reports of his brother during this (say) 50 year long journey (the star is about 50 light years far away from Earth). Time appears to run very slowly in these reports for the most of the trip duration because of time dilatation, so when the two brothers meet again on Earth, the traveler had aged only a few months in the 50 years that passed for his brother on Earth. However, the traveler also receives video reports from his brother on Earth. To the traveler Earth appears to move at the near-light-speed relative velocities, so he should observe a very slow passage on time on Earth. When the two brothers meet again, the guy staying on Earth should have aged negligibly in comparison to the traveler. Well, only one of these scenarios could happen. Which one? Surely we cannot expect the predictions of special relativity to work for the traveler because his frame of reference is not inertial (in order to turn around and come back to Earth, the traveler must carry out accelerated motion).

- The general theory of relativity overcomes all these difficulties and does not require ill-defined inertial reference frames.

Postulates

- Einstein formulated the general theory in the same fashion as the special one. The fundamental postulates are:

  1. The laws of nature must be the same in all reference frames (no preferred reference frame can be deduced from the equations of motion alone).
  2. Inertial forces in an accelerated frame of reference are fundamentally indistinguishable from gravity (equivalence principle)
  3. The principles of special theory of relativity hold on length and time scales where motion appears approximately uniform.

- The first postulate is the same as in the special relativity, except that now it applies to any reference frame, not necessarily inertial.

- The second postulate starts from a benign observation that gravitational force acting on an object is always proportional to its mass. For example, the gravitational force between two objects with masses M and m is according to the Newton's formula

$$\mathbf{F} = G\frac{Mm}{r^2}\hat{\mathbf{r}}$$

  where r is the distance between the two objects, $\hat{\mathbf{r}}$ the unit vector along the direction between the objects, and G is the gravitational constant. We could interpret this expression as $\mathbf{F} = m\mathbf{g}$ where $\mathbf{g}$ is the gravitational acceleration. If the body with mass M is very far away and very massive, all objects will experience the same gravitational acceleration $\mathbf{g}$ while the force on each object will be proportional to its mass. We could simulate gravity by a fast elevator. If the elevator accelerates with acceleration $\mathbf{a}$ and we set up a clock and a coordinate system in it, we will find that there is an inertial force acting on every object inside the elevator proportional to the object's mass and $-\mathbf{a}$. This looks just like gravity. Centrifugal force is another example, but this inertial force also has radial orientation, just like gravity from a nearby massive star. Astronauts inside the International Space Station can float around even though Earth pulls them down; it is the centrifugal force which cancels out the force of gravity, leaving behind the zero net force on astronauts.

- The equivalence principle is taken very seriously in the general theory of relativity. It provides the means to link the origin of dynamics, gravity, to its kinematic effect, accelerated motion. This is mathematically accomplished by constructing a curved Minkowski space-time. Dynamically, curvature is caused by massive objects in the universe. Kinematically, objects are regarded to always move freely by inertia, but in the curved space-time. It is this curvature that gives rise to acceleration. The mathematical description of curved space-time dynamics is well beyond the scope of this course.

- The third postulate is the backbone for the development of the formalism. It implies that if there are no massive sources of gravity around, light objects will move on straight lines and obey the rules of the special theory of relativity. Otherwise, we can obey the rules of the special theory only in very short time intervals during which an object moves along a short straight segment.

Some predictions

- Moving massive objects exert forces similar to magnetic forces in electrodynamics. Such forces are behind the intricate precession of Mercury's orbit in our solar system.

- Massive galaxies and stars can bend the trajectory of light and act like gravitational lenses. Multiple images of the same stars are observed in many places in the sky due to gravitational lensing.

- A black hole is a star whose gravitational field is so strong that light emitted from its surface must unavoidably fall back on it. Einsteins equations which summarize the relativistic physics of gravity predict in fact that there is a spherical boundary around the black hole, called event horizon, beyond which no information can be sent outside. Any object which finds itself inside the event horizon will inevitably fall onto the black hole in a finite amount of time, and no physical force can be generated to prevent that. The same applies to any substance at the black hole surface, there is no physical process which can sustain pressure strong enough to prevent the gravitational collapse onto the point at the center, the singularity. Hence, the general theory of relativity predicts that the black hole is a point of infinitely small volume, finite mass and infinitely large density. The radius of the event horizon is a function of the black hole mass, but the black hole cannot have any other properties (except angular momentum and charge).

- Strong gravitational fields cause time dilatation.

- Accelerating massive objects emit gravitational waves which can then propagate independently through space. A gravitational wave is an oscillatory wave-like stretching and contracting of the fabric of space-time, affecting local measurements of time intervals and length. There is an ongoing experiment attempting to detect gravitational waves produced in cataclysmic cosmic events (stars falling on black holes, etc.). However, gravitational waves are extremely weak.