Universal phase diagram of fermionic quantum liquids near the unitarity limit

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Motivation from recent experiments


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Field theory at unitarity

- Feshbach resonance
  - Unitarity = universality
  - RG Fixed point

- Field theories
  - One-channel approach
  - Two-channel approach
  - $\text{Sp}(N)$ generalization

Universal Phase Diagram of interacting quantum liquids near the unitarity limit

Predrag Nikolić
One channel model

- Critical field theory of interacting atoms

\[ S_c = \int d\tau d^d x \left\{ \psi^\dagger_\sigma \left( \frac{\partial}{\partial \tau} - \frac{\nabla^2}{2m} \right) \psi_\sigma + u_0 \psi^\dagger_\uparrow \psi^\dagger_\downarrow \psi^\dagger_\downarrow \psi^\dagger_\uparrow \right\} \]

- Relevant perturbations

\[ S_p = \int d\tau d^d x \left\{ -\mu \psi^\dagger_\sigma \psi_\sigma - h \left( \psi^\dagger_\uparrow \psi^\dagger_\uparrow - \psi^\dagger_\downarrow \psi^\dagger_\downarrow \right) \right\} \]

- Exact renormalization group
  - Perturbing about the vacuum (T=0, \(\mu=h=0\))
RG in one channel model

- Renormalization group flow
  \[ \frac{du}{d\ell} = (2 - d)u - \frac{u^2}{2} \]
  \[ \frac{d\mu}{d\ell} = 2\mu \quad \frac{dh}{d\ell} = 2h \]

- \( d < 2 \), repulsive interactions
  stable fixed point: universal (dilute) quantum liquid

- \( d > 2 \), attractive interactions
  unstable fixed point: Feshbach resonance
  Detuning from the Feshbach resonance: \( \nu \propto u - u^* \)
Two channel model

- Critical field theory of interacting atoms and molecules (s-wave)

\[ S_c = \int d\tau d^dx \left\{ \psi_\sigma^\dagger \left( \frac{\partial}{\partial \tau} - \frac{\nabla^2}{2m} \right) \psi_\sigma + \Phi^\dagger \left( \frac{\partial}{\partial \tau} - \frac{\nabla^2}{4m} + r_0c \right) \Phi \\
- g_0 \left( \Phi^\dagger \psi_\uparrow \psi_\downarrow + \Phi \psi_\downarrow^\dagger \psi_\uparrow^\dagger \right) \right\} \]

- Relevant perturbations

\[ S_p = \int d\tau d^dx \left\{ -\mu (\psi_\sigma^\dagger \psi_\sigma + 2\Phi^\dagger \Phi) + \delta \Phi^\dagger \Phi - h \left( \psi_\uparrow^\dagger \psi_\uparrow - \psi_\downarrow^\dagger \psi_\downarrow \right) \right\} \]

\[ \delta = \frac{m\nu}{4\pi} g_0^2 \]
Feshbach resonance

- Exact renormalization group

  \[
  \frac{d g}{d \ell} = \frac{(4 - d)}{2} g - \frac{g^3}{2}
  \]

  \[
  \frac{d \mu}{d \ell} = 2\mu \quad \frac{d h}{d \ell} = 2h \quad \frac{d \nu}{d \ell} = (2 - g^2)\nu
  \]

- Renormalization group flow

- The critical exponents of the relevant operators are the same as in the one-channel model

  **Both models describe the same fixed point** => Feshbach resonance
**ε-Expansions**

- Thermodynamic properties are universal functions at the resonance
  - Expressed as expansions in powers of $\varepsilon$
  - One-channel: $\varepsilon = d - 2$; two-channel: $\varepsilon = 4 - d$

- Fundamental limitations
  - Perturbation theory in the interaction strength $(u_0, g_0)$
  - Critical coupling must be small $(u_0, g_0 \sim \varepsilon)$
  - Tractable effects of pairing fluctuations are small
    - One-channel $\Rightarrow$ negligible pairing amplitude
    - Two-channel $\Rightarrow$ deeply in the superfluid, or 1 Fermi sea

- Large-$N$ expansion overcomes these limitations
Large-$N$ expansion

- Sp($2N$) generalization of the two-channel model
- $N$-pairs of spin-up and spin-down fermions, $\psi_\sigma$ ($\sigma=1..2N$)
- Fermions coupled to a single molecule field, $\Phi$

\[
S_c = \int d\tau d^d x \left\{ \psi_\sigma^\dagger \left( \frac{\partial}{\partial \tau} - \frac{\nabla^2}{2m} \right) \psi_\sigma + \Phi^\dagger \left( \frac{\partial}{\partial \tau} - \frac{\nabla^2}{4m} + r_{0c} \right) \Phi \\
- \frac{g_0}{2} \left( \Phi^\dagger \mathcal{J}^{\alpha\beta} \psi_\alpha \psi_\beta + \text{H.c.} \right) \right\}
\]

\[
\mathcal{J} = \begin{pmatrix}
    & 1 & \\
-1 & 1 & \\
-1 & -1 & \\
    & \ddots & \ddots
\end{pmatrix}
\]
Relevant perturbations

\[ S_p = \int d\tau d^d x \left\{ -\mu (\psi_\sigma^\dagger \psi_\sigma + 2\Phi^\dagger \Phi) + \delta \Phi^\dagger \Phi + h \sum_\sigma (-1)^\sigma \psi_\sigma^\dagger \psi_\sigma \right\} \]

- \( \mu \) ... chemical potential
- \( h \) ... Zeeman field
- \( \nu \) ... detuning from the Feshbach resonance

\[ \delta = \frac{m \nu}{4\pi} Ng_0^2 \]

Renormalization group flow:

\[ \frac{d\mu}{d\ell} = 2\mu \]

\[ \frac{dg}{d\ell} = \frac{(4 - d)}{2} g - Ng^3 \]

\[ \frac{dh}{d\ell} = 2h \]

\[ g^* = \text{const.} \times g_0^* \sim \frac{1}{\sqrt{N}} \]
Universal mean-field phase diagram

- Free energy of molecules
  - Static uniform superfluid (mean-field $\Phi=$const. at $N=\infty$)
  - Integrate-out fermions (Gaussian)

\[
\frac{\mathcal{F}}{N} = \frac{m\nu}{4\pi} |\Phi|^2 - \int \frac{d^3p}{(2\pi)^3} \left[ \sqrt{\left( \frac{p^2}{2m} - \mu \right)^2 + |\Phi|^2} - \left( \frac{p^2}{2m} - \mu \right) - \frac{m}{p^2} |\Phi|^2 \right] \\
+ T \ln \left( 1 + e^{-\left( \sqrt{\left( p^2/(2m) - \mu \right)^2 + |\Phi|^2} - h \right)/T} \right) \\
+ T \ln \left( 1 + e^{-\left( \sqrt{\left( p^2/(2m) - \mu \right)^2 + |\Phi|^2} + h \right)/T} \right)
\]

- Obtain $|\Phi|$ by minimizing $\mathcal{F}(|\Phi|)$
- Substitute $|\Phi|$ in the fermion action to check the Fermi seas
Uniform $T=0$ phases and transitions

$h_c(0) = 0.807 \mu + O(1/N)$

**Predictions**
- 1$^{\text{st}}$ order transition between the superfluid and normal phases
- Smooth BEC-BCS crossover
- Uniform magnetized BEC superfluid phase for $\mu<0$
- Normal phases with one ($1N$) or two ($2N$) Fermi seas
Other views of the $T=0$ phase diagram

- The number of Fermi seas is trivially decided by $h$ in the absence of superfluid (mean-field)

- Universal phase diagram in proximity of the RG fixed point
Equation of state: Normal phase at $T=0$

\[
\frac{P}{N} = \frac{\mu (2m\mu)^{3/2}}{15\pi^2} \left[ \theta \left( 1 - \frac{\hbar}{\mu} \right) \left( 1 - \frac{\hbar}{\mu} \right)^{3/2} + \theta \left( 1 + \frac{\hbar}{\mu} \right) \left( 1 + \frac{\hbar}{\mu} \right)^{3/2} \right] + \frac{\delta P}{N}
\]

\[
\frac{\delta P}{N} = \mu (2m\mu)^{3/2} F_{\delta P} \left( \frac{\hbar}{\mu}, \frac{\nu}{\sqrt{2m\mu}} \right)
\]

- Pairing fluctuations increase pressure
- Longer-lived pairs $\Rightarrow$ larger pressure
- No pressure increase without two Fermi seas
Finite temperatures

- 2\textsuperscript{nd} order superfluid-normal phase transition at $T=T_c$

$$\frac{\mu}{T_c} = 1.50448 + \frac{2.785}{N} + \mathcal{O}(1/N^2)$$

$$\frac{\varepsilon_F}{T_c} = 2.01424 + \frac{5.317}{N} + \mathcal{O}(1/N^2)$$

$$\frac{P/N}{(2m)^{3/2}T_c^{5/2}} = 0.13188 + \frac{0.4046}{N} + \mathcal{O}(1/N^2)$$

Monte-Carlo: E.Burovski, N.Prokof'ev, B.Svistunov, M.Troyer

Conclusions

- Theory of unitarity in fermionic quantum liquids
  - Interacting atoms: one-channel model ($\varepsilon=d-2$)
  - Interacting atoms and molecules: two-channel model ($\varepsilon=4-d$)
  - Sp($N$) generalization

- Universal phase diagram
  - Feshbach resonance is an RG fixed point
  - Uniform superfluid and normal phases
  - BEC-BCS crossover
  - 1$\text{st}$ order superfluid-normal transition by population imbalance
  - Exotic magnetized uniform superfluid at $T=0$
  - $1/N$ corrections to the equation of state in the $T=0$ normal phase
  - Universal ratios of observables at $T>0$ phase transition