The velocity of propagation of electromagnetic waves derived from the resonant frequencies of a cylindrical cavity resonator

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The frequency of resonance of an evacuated cavity resonator in the form of a right circular cylinder is given by the formula

\[ f = \nu_0 \sqrt{\left[ \frac{r^2}{\pi D^2} + \frac{n^2}{2L} \right] \left[ 1 - \frac{1}{2Q} \right]}, \]

in which \( \nu_0 \) is the free-space velocity of electromagnetic waves, \( D \) and \( L \) are the internal diameter and length respectively of the cylinder, \( r \) is a constant for a particular mode of resonance, \( n \) is the number of half-wave-lengths in the resonator and \( Q \) is the quality factor. Assuming the validity of this equation the value of \( \nu_0 \) can be obtained from measured values of \( f, D, L \) and \( Q \).

A copper cylinder of diameter approximately 7.4 cm. and length 8.5 cm. was constructed with the greatest uniformity of diameter and squareness of end-faces and its dimensions were measured. The resonant frequencies for a number of different modes were measured and experiments were made to show that the effects on frequency of the coupling probes to the oscillator and detector were negligibly small. It was concluded from these measurements that the most favourable experimental conditions can be obtained for the \( E_{010} \) and \( E_{011} \) modes. Final measurements on these gave

\[ \nu_0 = 299,792 \text{ km./sec.} \]

The estimated maximum error of the result is 9 km./sec. (3 parts in 10^6). This is the error of a single measurement and, since most of the errors are not necessarily random, little is gained by making a large number of measurements. The value is 16 km./sec. greater than the recently determined values of the velocity of light, although the results are not in disagreement when the combined limits of accuracy are taken into account.

1. INTRODUCTION

The velocity of propagation of electromagnetic waves is, in Maxwell's theory, given by \( 1/(\mu \varepsilon) \), \( \mu \) being the permeability and \( \varepsilon \) the permittivity of the medium in rationalized practical units. The value for a vacuum, \( 1/\sqrt{\mu_0 \varepsilon_0} \), is a constant of fundamental importance, being theoretically identical with the velocity of light, and appearing in a great number of electrical calculations. During recent years its value has also become of significance in the field of radio engineering, since it is required for the calculation of the resonant frequencies of electrical circuits, such as cavity resonators, and also for methods of navigation depending on radio waves. The measurement of this constant has attracted many research workers, and possibly more effort has been devoted to it than to the determination of any other general constant. In most of the experiments that have been performed the velocity of light has been determined directly by the measurement of the time occupied in its travel over a certain distance. In all cases the time interval is produced by a revolving or alternating mechanism so that the quantities actually measured are frequency and
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distance. Both of these can be measured with a high accuracy—better than 1 part in \(10^5\) and the main experimental difficulty appears to be the adjustment of the variable frequency or length to correspond to the fixed length or frequency as the case may be. In order to eliminate as far as possible the random observational errors a great number of determinations are necessary. Even so it was found in the two most recent experiments (Michelson, Pease & Pearson 1935; Anderson 1941) that the means of groups of a considerable number of measurements differed by amounts greater than would be expected if the errors were entirely random. Michelson et al. found differences of 11 km./sec. between the means of groups consisting of about 500 measurements and a maximum difference of 93 km./sec. between the means of groups of about six measurements. Anderson does not give the individual results, but he obtained a difference of 67 km./sec. between the means of groups of about 200 measurements. The close agreement between their final results of 299,774 ± 11 and 299,776 ± 14 km./sec. may therefore be misleading.

The constant has also been determined by a comparison between the calculated and measured values of the capacitance of a capacitor of suitable shape (Rosa & Dorsey 1907). The value calculated from electromagnetic theory involves the permittivity of the medium and the dimensions of the capacitor. The measured value, obtained by balancing a bridge network of which the capacitor forms one arm, is expressed in terms of the international ohm. The experiment consists in the measurement of the dimensions of the capacitor, and of a resistance, a resistance ratio, and a frequency, and the result can be interpreted as the measured value of the permittivity of free space. From this the velocity can be calculated, the value of permeability being known by definition. Both the calculations and experiments are beset with difficulties, but the result obtained by Rosa & Dorsey was probably the most reliable up to that time. The final value given is 299,710 km./sec., which is the mean of about 900 individual determinations, the average deviation is 22 km./sec. and the estimated maximum error is 30 km./sec., apart from uncertainties in the value taken for the international ohm. This latter constant is now known with an accuracy of about 2 parts in \(10^5\), and Birge (1941) has applied a correction to their results to obtain a value of 299,784 km./sec. Using all the reliable data existing at the time Birge gives the most probable value for the constant as

\[
c = 299,776 \pm 4 \text{ km./sec.}
\]

In recent years a number of experimenters have measured the speed of travel of electric waves of frequencies of the order of a few Mcyc./sec., but the accuracy achieved was in general inferior to 1 % (Smith-Rose 1943), so that these measurements cannot yet be taken into account in discussions on the value of the constant.*

The experiment described in this paper presents fewer technical difficulties than any method previously described. It consists in the measurement of the frequency of electrical resonance of a hollow copper cylinder, and the calculation of the frequency from electromagnetic theory. It is similar in principle, therefore, to the experiment of Rosa & Dorsey but also has some points in common with the

* The accuracy has been greatly increased in works published since this was written and the probable error is now of the order of 1 part in \(10^4\) (Jones 1947; Smith, Franklin & Whiting 1947).
velocity of light experiments in that the only quantities measured are the dimensions of the apparatus and a frequency. The frequency measured is, however, that of the electromagnetic wave itself and not that of a modulation imposed on the wave. Moreover, the frequency of the oscillations is of the order of $3 \times 10^9$ cyc./sec. compared with $5 \times 10^{14}$ cyc./sec. for light vibrations. The dimensions and the frequency can be aligned by setting to resonance with a precision of a few parts in $10^6$ so that the accuracy of observation is of the same order as that of the measurement of the dimensions and frequency. Little is gained, therefore, by making a large number of determinations, and the authors feel justified in claiming a rather higher accuracy for a single measurement than is claimed by previous workers for their averaged results. The result obtained is 299,792 km./sec. This value is 1 km./sec. lower than that which has already appeared in Essen (1947), but the difference arises only from the fact that as the work proceeded the calculations were made to more significant figures than appeared to be warranted at first.

The experiment arose out of work on cavity resonator wave-meters, and some deductions concerning the velocity of propagation were made in a previous paper (Essen 1946). The work has been done concurrently with an already full programme, and except for the cavity resonator itself existing equipment has been used. As the experiment progressed it became clear that further work with different modes of resonance and with cavities of different types would be of interest, particularly as the final result obtained is 16 km./sec. higher than the previously accepted value of 299,776 km./sec. Such work would provide further experimental evidence concerning the extent of the different sources of possible error.

Plans have been made to continue the work as the opportunity arises, but in the meantime it is thought that the measurements already made are of sufficient interest to merit publication.

2. THEOREY OF THE METHOD

The resonant frequency of a right circular hollow cylinder closed at both ends is given by Sarbacher & Edson (1943) as

$$f_{lmn} = \sqrt{\left(\frac{r}{\pi D}\right)^2 + \left(\frac{n}{2L}\right)^2}$$

in which $f_{lmn}$ is the frequency in cyc./sec. of the mode designated by the suffix, $v$ is the velocity of propagation in cm./sec. in the medium ($v = 1/\sqrt{\mu \epsilon}$), $D$ is the diameter and $L$ the length of the cylinder in cm., $n$ is a whole number having the same value as the third suffix in the mode designation and $r$ is the mth root of the Bessel equations $J_0(x) = 0$ for $E$ modes and of $J_1(x) = 0$ for $H$ modes. If the cavity is evacuated $\mu = \mu_0$ and $\epsilon = \epsilon_0$ and the velocity is that for free space. We have then

$$v_0 = \sqrt{\left(\frac{r}{\pi D}\right)^2 + \left(\frac{n}{2L}\right)^2}$$

The modes of resonance which were finally used in the experiment were the $E_{010}$ and $E_{011}$ and for both of these modes the value of $r$ is 2.404825, $n$ being 0 and 1 respectively.
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The above formula is derived on the assumption that the walls of the cavity are perfectly conducting. In practice, owing to the finite conductivity of the walls, the magnetic field penetrates them to some depth, and the field configuration therefore differs from that assumed in the case of a perfect resonator. No complete solution has yet been obtained for this case, but analogy with electrical circuits resonating at low frequencies suggests that the effective inductance of the system is increased as a result of the field inside the walls and that the resonant frequency is therefore reduced. Approximate expressions have been derived for the effect on the resonant frequency. Bernier (1946), for example, has shown that if the propagation constant in the lossless guide is \( \beta = 2\pi/\lambda = 2\pi f/v \), where \( \lambda \) is the wave-length in the guide, then the propagation constant in a guide of finite conductivity is

\[
\beta' = \beta \left[ 1 - \frac{\mu eA}{2 \sqrt{2}} (1 - j) \right],
\]

where \( A \) is a function of the field and \( e \) is the skin depth which depends on the conductivity in accordance with the well-known skin-effect formulae. The imaginary part of this expression corresponds to the attenuation, but the real part corresponds to a reduction in frequency. Putting the expression in terms of frequency we have

\[
f' = f \left[ 1 - \frac{\mu eA}{2 \sqrt{2}} (1 - j) \right],
\]

and the frequency measured will be the real part of this expression, \( f \left( 1 - \frac{\mu eA}{2 \sqrt{2}} \right) \).

This can be expressed in terms of the \( Q \) of the resonator which can be measured experimentally, because

\[
Q = \frac{\text{real } \beta'}{2 \text{ imag. } \beta'} = \frac{1 - (\mu eA/2 \sqrt{2})}{\mu eA/2 \sqrt{2}},
\]

and therefore

\[
f \left( 1 - \frac{\mu eA}{2 \sqrt{2}} \right) = f \left( 1 + \frac{1}{2Q} \right),
\]

\((\mu eA/2 \sqrt{2})\) being a very small quantity, higher powers of which have been neglected.

For \( v_0 \) we must therefore use in place of (2)

\[
v_0 = \frac{f'_{lmn} \left( 1 + 1/2Q \right)}{\sqrt{(r/nD)^2 + (n/2L)^2}},
\]

\( f'_{lmn} \) being the measured frequency and \( Q \) the measured quality factor.

The measured \( Q \)’s were rather lower than the calculated values, which is a general experience with cavity resonators. The reduction is probably due to either mechanical imperfections of the surface or an increased resistivity of the skin due to the mechanical treatment it receives. In any case it seems likely that the effect causing higher resistivity must also give a greater frequency correction and that the measured value should be used.

The resonator is coupled to the source of oscillations and to a detector by small probes through holes in one of the end-walls of the cylinder, and it is important to consider the effects of this coupling on frequency. The theoretical treatment would present considerable difficulties, but previous experimental results had indicated
that the degree of coupling could be reduced until the effect on frequency was negligible. It was decided, therefore, to treat this as an experimental problem and to establish that the effect was negligible within prescribed limits of accuracy.

3. Description of the apparatus

3.1. General arrangement

The apparatus is shown diagrammatically in figure 1. The cavity resonator is coupled to an oscillator and a receiver by means of the probes A and B connected to coaxial lines. The frequency of the oscillator is varied, and when it reaches a value corresponding to one of the resonant frequencies of the cavity there is a sharp increase in the amplitude of the signal, indicated on a meter in the receiver. The frequency is set to give a maximum indication and its value is then measured by means of a heterodyne wave-meter. The resonator was enclosed under a bell-jar which could be evacuated, a free air path being left in the plug and socket fittings at A and B.

![Figure 1. Schematic representation of the experiment.](image)

3.2. The cavity resonator

A number of factors were considered in the choice of the material, shape and size of the resonator. In order to obtain a sufficiently sharp resonance the inner surface, to a depth equal to the penetration of the field, must be highly conducting, but unfortunately highly conducting metals such as copper and silver are also soft and have a high temperature coefficient of expansion. They cannot therefore be worked with the highest precision, and the measurement of the dimensions of a resonator made from such materials presents some difficulty. The alternatives of plating a hard metal or fused quartz were considered, but it was thought that the plating could not be effected with the necessary precision and that the soft metal surface would still require to be worked. Most of the advantages would therefore be annulled, and the remaining one of low-temperature coefficient was not believed to be of paramount importance as temperature was unlikely to be a limiting factor in
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The velocity of propagation of electromagnetic waves has an accuracy of measurement. It was decided, therefore, to use solid copper as the material at least for the first experiments.

The frequencies of resonance can be calculated readily by existing theoretical methods for only simple shapes of resonator, such as the right circular hollow cylinder, the sphere and the rectangular parallelepiped.

Of these the last has some advantages, since it can be built from optically flat plates and its dimensions measured by interferometer methods, but it would be more difficult to make than the cylinder, and the large number of sharp edges and joints is a disadvantage. The sphere has the advantage of freedom from sharp edges but is not easy to make with the required precision. It was decided, therefore, to use the cylindrical form for the first experiments. There is considerable latitude in the choice of the dimensions of the resonator and the resonant frequency. Oscillators are available for any frequency up to 10,000 Mcy./sec. and the skin-effect correction decreases with increase of frequency. On the other hand, the power and frequency stability of the oscillators both decrease with increase of frequency, particularly at values much higher than 3000 Mcy./sec. The optimum size of the cylinder from the point of view of precision construction, with existing honing equipment, was of the order of 8 cm. both in diameter and length. Such a cylinder gives low-order resonances in the region of 3000 Mcy./sec., and it appeared therefore that this was the best frequency to use. The exact diameter and length were then chosen so that several low-order modes of resonance could be obtained in this region of frequency but spaced sufficiently to be regarded as single resonances in the theoretical treatment.

Details of the resonator are shown in figure 2. The cylinder is turned from the solid and then honed to be as uniform in dimensions as possible, the exact size not being important. The end-faces are ground square and parallel. The end-plates are ground flat and fixed to the cylinder by eight screws, an ample clearance being allowed in the holes in the plates. The coupling probes A, B pass through holes in one of the end-plates, the depth of insertion being adjustable. In the final condition no. 38 s.w.g. wire (diameter 0.0060 in. = 0.0152 cm.) was used through holes of 0.03 cm. diameter. Probe A is connected to a socket in which is plugged the coaxial lead to the oscillator. If the superheterodyne receiver is used probe B is terminated in a similar socket, but if the detecting system is a crystal detector and galvanometer the crystal is connected between the probe and the socket as shown at C in the diagram. The metal piece D acts as a radio-frequency by-pass condenser and the galvanometer is plugged into the socket E. The circuit requires a direct current return lead which is provided by a wire F, the position of which can be altered to tune the crystal circuit. The insulating material G and the sockets are drilled to permit the free passage of air during the evacuation of the system.

3.3. The receiver

The crystal detector used in the early experiments was a standard cartridge mounted silicon crystal with a tungsten wire (type no. CV 113), and its direct current output was observed on a Tinsley type 4500 A (50 ohms) galvanometer. For the final measurements it was decided to use instead a superheterodyne receiver which was available and which proved to be slightly more sensitive. This is a special form of
instrument (known as a spectrum analyzer or spectrometer) developed for observing
the spectral distribution of energy in the output of transmitting valves. Its useful
feature in the present application is that its sensitivity remains almost constant,
without adjustment of any controls, over a small band of frequencies. The tuning
controls of the receiver are set so that the resonant frequency of the cavity is near
the middle of this band. Then as the oscillator frequency is varied through the
resonant value a peak deflexion is observed without any retuning of the receiver.

![Diagram of resonator](image)

**Figure 2. Details of the resonator.**

3·4. *The heterodyne wave-meter*

This instrument and its method of use have already been described by the authors
(1945). With careful operation it enables frequencies in the region concerned to be
measured with an accuracy of ± 2 parts in 10⁶.

3·5. *The oscillator*

The oscillator valves were chosen from those used as local oscillators in receivers,
the type CV 35 being suitable for the $E_{010}$ mode and the CV 234 for the $E_{011}$ mode.
The former was operated from a stabilized mains supply unit and the latter from
batteries. The tuning mechanisms supplied with the valves were found to give a
sufficiently smooth frequency variation, although it was sometimes necessary to
choose a good portion of the range of the tuning mechanism by the appropriate
adjustment of subsidiary tuners.
4. MEASUREMENT OF RESONANT FREQUENCIES AND QUALITY FACTORS

The assembly of resonator and bell-jar was well lagged to ensure that any temperature changes occurring were very slow, and that the value recorded on a thermometer in good thermal contact with the resonator represented its actual temperature within 0.1° C. The system was evacuated at least 12 hr. before the electrical measurements were made. The electrical equipment was also switched on some hours before it was required for use in order that the various oscillators would reach a steady temperature and the measurements could be made under the most favourable conditions.

As explained in § 3.1 the measurement consisted in the adjustment of the oscillator to the resonant frequency of the cavity and then in the measurement of this frequency in terms of the standard. It was carried out by two observers to eliminate any appreciable time interval between the two operations. Precautions were taken to ensure that the maximum deflexion in the receiver did in fact correspond to the resonant condition of the cavity.

It was established that there was no appreciable change in the output of the oscillator as the tuning was varied in the region of resonance. There are in the circuit several short lengths of cable and connecting plugs and sockets. The impedance of these and therefore the power available at the probes changes with frequency, but owing to the resistance of the cable such tuning effects were of a different order of sharpness from the cavity resonance. It was easy to arrange that for the small frequency change involved in passing through resonance there was no appreciable amplitude change due to these effects. An overall check was obtained, in preliminary experiments, by increasing the insertion of the probes in order to give a convenient deflexion on the oscilloscope of the receiver by direct coupling between them when the cavity was not in resonance. The frequency was varied a few megacycles on both sides of the resonant value, and it was verified that the amplitude of the oscilloscope deflexion remains constant as it moves along the time base on both sides of the cavity resonance.

The voltage supplies to the oscillator were set so that frequency variations due to any residual fluctuations in them were reduced to a minimum. Under good conditions the frequency band over which an audible beat note could be obtained in the heterodyne wave-meter was less than 1 part in 10⁵ of the frequency, and the setting to the mid-point of the band was made with a precision of ± 2 parts in 10⁶.

The Q of the resonator was measured by observing the change in frequency from the resonant value required to reduce the detected current to 1/\sqrt{2} of its peak value. If this change is \( \delta f \) then the \( Q \) is \( f/2\delta f \). The value was checked by measurements made with the crystal detector and galvanometer under conditions for which the detector was known to give a square law within close limits.

5. MEASUREMENT OF DIMENSIONS

The dimensions of the resonator were measured in the Metrology Division of the National Physical Laboratory.
5.1. Internal diameter

The measurements were made in a temperature-controlled room with a horizontal comparator arranged for internal measurements through steel contact tips of rounded form (about 0.4 cm. radius) operating under a measuring force equivalent to 0.34 kg. weight. A suitably wrung combination of slip-gauges and end-pieces served as the reference basis for internal comparison. Differences between this and any particular diameter of the tube were indicated on the optical scale of the comparator which could be read directly to 1.3 μ and by estimation to 0.3 μ (0.00001 in.). The measurements were made over eight symmetrically disposed diameters at seven positions along the axis of the cylinder.

The accuracy of determination of the mean measured diameter was estimated at ±1.3 μ for the preliminary measurements which were carried out as normal routine tests. To improve this for the final measurement the following modifications were made in the technique of measurement:

(a) The comparator was screened by means of a glass panel so that temperature effects due to the presence of the observer were minimized as much as possible.

(b) An improved form of reference standard was used. This was of the ‘box’ type and consisted of a firmly wrung combination of two piles of high-quality slip-gauges not differing in length by more than about 0.03 μ, between two lapped end-pieces, thus constituting a rigid ‘box’, or frame for which the reference dimension was well established. Each slip-gauge employed in the combination had previously been calibrated by interferometer measurements in terms of wave-lengths of light to an accuracy equivalent to 0.03 μ.

(c) Two diameters at the centre of the copper cylinder and in directions at right angles to one another were accurately measured in terms of the reference standard.

(d) The temperatures of the copper cylinder and the hardened steel standard were observed and the results of the intercomparison were reduced to 20° C, using thermal coefficients of expansion of 12 × 10⁻⁶ and 17 × 10⁻⁶ for hardened steel and copper respectively.

(e) Having established two basic diameters all other diameters of the cylinder were compared with first one and then the other of the basic diameters. The effect of any temperature variations on the results obtained during each of these long series of comparisons was thereby eliminated provided any changes in temperature were reasonably the same for all parts of the cylinder. The mean measured diameter derived from each of the two series agreed to 0.5 μ.

(f) Theoretical and experimental investigations were made of the different effects of the compressive force exerted by the steel measuring tips of the comparator on hardened steel and copper surfaces. The results obtained by calculation and experiment agreed to 0.03 μ and showed that the difference could be compensated by reducing the observed value of internal diameter of the cylinder as derived from the standard by 0.27 μ.

5.2. Length

Length measurements were made by means of a vertical comparator having a rounded contact measuring tip and an optical scale similar to that of the horizontal comparator. One end of the cylinder was placed in contact with the horizontal platen.
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of the comparator under the contact tip, and the height of the other end of the cylinder was measured at various positions by comparison with a wrung combination of reference standard slip-gauges. Under these conditions of measurement it was considered that the compressive effect of the single contact tip was negligible in relation to the general accuracy of the length determination.

The overall length of the tube was thus measured at eight positions symmetrically spaced around the cylinder, and at each of these positions three measurements were made, one near the bore, one at the mid-point of the wall thickness and one near the outside of the tube. The measurements are considered to be reliable to ±0·8μ.

Some tests were made to show whether the diameter and length were affected by screwing on the end-plates, but within the accuracy of these measurements (±0·8μ) there was no evidence of deformation.

6. Experimental results

6.1. Preliminary measurements

The method of coupling used favours the excitation of the $E_{01n}$ modes but the $H_{01l}$ and $H_{11l}$ modes can be excited if the probes are bent round to form small loops orientated so as to give a linkage with the field associated with the particular mode. To gain experience of the most suitable experimental conditions preliminary measurements were therefore made of the resonant frequencies of a number of different modes.

From the frequencies the values for velocity were calculated and the results are included in Table 1, but they are not considered to be more accurate than 6 parts in $10^5$ for the following reasons:

(a) The metrological examination of the cylinder revealed larger variations of diameter than anticipated, and the measurements themselves were not made with the maximum possible accuracy, the uncertainties being given at the head of Table 1. The end-faces were not ground flat and the length measurement is therefore averaged to allow for this, the deviations being of the order of ±5μ.

(b) The conditions for the frequency measurement were not ideal. For the $E_{012}$, $H_{01l}$ and $H_{11l}$ modes the coupling probes were inserted about 1 or 2 mm., and in the case of the two latter modes were formed into small loops. The size of the coupling holes was 0·15 cm.

### Table 1. (Preliminary measurements.) Resonant frequencies of the copper cylinder and calculated values of velocity

<table>
<thead>
<tr>
<th>mode of correction factor</th>
<th>correction factor</th>
<th>constant $r$</th>
<th>frequency $f^*$ (Mcyc./sec.)</th>
<th>velocity $v_0$ (km/sec.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{010}$</td>
<td>1·000028</td>
<td>2·404825</td>
<td>3102·12</td>
<td>299,797</td>
</tr>
<tr>
<td>$E_{011}$</td>
<td>1·000035</td>
<td>2·404825</td>
<td>3562·38</td>
<td>299,798</td>
</tr>
<tr>
<td>$E_{012}$</td>
<td>1·000030</td>
<td>2·404825</td>
<td>4378·81</td>
<td>299,785</td>
</tr>
<tr>
<td>$H_{011}$</td>
<td>1·000015</td>
<td>3·831706</td>
<td>5343·96</td>
<td>299,799</td>
</tr>
<tr>
<td>$H_{111}$</td>
<td>1·000031</td>
<td>1·841184</td>
<td>2950·78</td>
<td>299,777</td>
</tr>
</tbody>
</table>

Mean diameter of resonator 7·39759 ± 0·00012 cm. Variation ± 0·00045 cm.

Length of resonator 8·55844 ± 0·0003 cm.
(c) As a result of these measurements it was decided to concentrate on the $E_{010}$ and $E_{011}$ modes, which could be detected readily with the probes flush with the wall of the cavity. The size of coupling hole was reduced. It was at the same time decided to regrind the cylinder to improve its uniformity of diameter and length.

6-2. *Effect of size of coupling hole and the penetration of the probes*

It was necessary to establish experimentally the effect on frequency of the presence of the coupling holes and probes. First measurements were made with three sizes of coupling hole, the probes being in each case as nearly flush as possible to give an adequate indication. The results are given in table 2. It was concluded from these results that any residual effect on frequency was less than 0·01 Mcyc./sec., i.e. less than 3 parts in $10^6$.

<table>
<thead>
<tr>
<th>mode</th>
<th>size of coupling hole (cm.)</th>
<th>insertion of probe (cm.) (estimated by eye)</th>
<th>resonant frequency (Myc./sec.)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E_{010}$</td>
<td>0·03</td>
<td>&lt;0·01</td>
<td>3101·246</td>
</tr>
<tr>
<td></td>
<td>0·07</td>
<td>flush</td>
<td>3101·251</td>
</tr>
<tr>
<td></td>
<td>0·16</td>
<td>flush</td>
<td>3101·242</td>
</tr>
<tr>
<td>$E_{011}$</td>
<td>0·03</td>
<td>&lt;0·01</td>
<td>3563·798</td>
</tr>
<tr>
<td></td>
<td>0·07</td>
<td>flush</td>
<td>3563·787</td>
</tr>
<tr>
<td></td>
<td>0·16</td>
<td>flush</td>
<td>3563·71</td>
</tr>
</tbody>
</table>

*Figures 3 and 4.* Effects of probe intrusion. Diam. of holes 0·03 cm. Diam. of probes 0·015 cm.
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The effect of probe insertion is shown in figures 3 and 4, and from these results it is concluded that the residual effect of the probes is again less than 0.01 Mcyc./sec.

The Q of the resonator for the two modes had already reached its maximum value with the coupling considerably closer than that finally used. The values are 18,000 and 14,000 respectively for the $E_{010}$ and $E_{011}$ modes as compared with the theoretical values of 21,000 and 17,600. The ratios of experimental to theoretical values are as great as those usually obtained for cavity resonators.

6.3. Final measurements

The final results for the velocity are based on a series of metrological measurements made between two series of electrical measurements. The detailed dimensions are given in tables 3 and 4. The cylinder is seen to be slightly oval in shape, the difference between the maximum and minimum diameters being $2\mu$ (3 parts in 10). Since the field for the modes employed is symmetrical about the axis, the effective diameter was taken as the mean value. The length at the inner edge of the cylinder was slightly greater than that at the outer edge and varied smoothly round the circumference by a total of $2.2\mu$. The effective length was therefore taken as the mean value at the inner edge.

**Table 3. Diameter of the resonator expressed as the difference from the mean value of 7.39957 cm. Unit 0.00001 cm. (0.1\mu)**

<table>
<thead>
<tr>
<th>axial position of measurement from one end (cm.)</th>
<th>measured diameter at 20° C in different diametrical planes</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.13</td>
<td>-15 -6 +1 +1 +7 +3 0 -10</td>
</tr>
<tr>
<td>1.3</td>
<td>-20 -11 -4 -2 +1 +2 -4 -16</td>
</tr>
<tr>
<td>2.8</td>
<td>-20 -13 -17 -2 +9 -7 -4 -20</td>
</tr>
<tr>
<td>4.3</td>
<td>-15 -9 -6 +3 +8 +10 -3 -14</td>
</tr>
<tr>
<td>5.8</td>
<td>-2 +6 +13 +13 +27 +24 +13 0</td>
</tr>
<tr>
<td>7.3</td>
<td>-3 -3 +6 +10 +14 +9 +7 0</td>
</tr>
<tr>
<td>8.4</td>
<td>+7 -1 +5 +8 +10 +8 +2 +9</td>
</tr>
</tbody>
</table>

**Table 4. Length of the resonator expressed as the difference from 8.53637 cm. (Mean of inner edge values). Unit 0.00001 cm. (0.1\mu)**

<table>
<thead>
<tr>
<th>position of measurement of near inner edge centre near outer edge</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 -8 -13 -16</td>
</tr>
<tr>
<td>2 +5 +2 -8</td>
</tr>
<tr>
<td>3 +9 +6 +1</td>
</tr>
<tr>
<td>4 +9 +3 +3</td>
</tr>
<tr>
<td>5 +7 +2 -3</td>
</tr>
<tr>
<td>6 -2 -5 -12</td>
</tr>
<tr>
<td>7 -13 -14 -17</td>
</tr>
<tr>
<td>8 -9 -13 -17</td>
</tr>
</tbody>
</table>
The measured values of resonant frequency and the values of velocity calculated from the resonant frequencies and dimensions are given in table 5. The frequencies were measured using the minimum coupling conditions as follows:

- Size of coupling holes: 0.03 cm.
- Diameter of probes: 0.015 cm.
- Insertion of probes: < 0.01 cm.

Resonance was detected on the spectrum analyzer as described in §3-3.

**TABLE 5. FINAL VALUES FOR RESONANT FREQUENCIES OF THE \( E_{010} \) AND \( E_{011} \) MODES AND THE DEDUCED VALUES OF THE VELOCITY OF PROPAGATION**

<table>
<thead>
<tr>
<th>Date</th>
<th>Mode of Resonance</th>
<th>Frequency Correction Factor ((1 + 1/2Q))</th>
<th>Constant (r)</th>
<th>Measured Frequency (f'_{\text{meas}}) (MHz)</th>
<th>Velocity (v_0) (km/sec)</th>
</tr>
</thead>
<tbody>
<tr>
<td>2. x. 46</td>
<td>( E_{010} )</td>
<td>1.000028</td>
<td>2.404825</td>
<td>3101.25</td>
<td>299,793</td>
</tr>
<tr>
<td>2. x. 46</td>
<td>( E_{011} )</td>
<td>1.000035</td>
<td>2.404825</td>
<td>3563.80</td>
<td>299,791</td>
</tr>
<tr>
<td>25. x. 46</td>
<td>( E_{010} )</td>
<td>1.000028</td>
<td>2.404825</td>
<td>3101.28</td>
<td>299,796</td>
</tr>
<tr>
<td>25. x. 46</td>
<td>( E_{011} )</td>
<td>1.000035</td>
<td>2.404825</td>
<td>3563.77</td>
<td>299,789</td>
</tr>
</tbody>
</table>

The estimated maximum errors due to various causes are listed below:

1. Setting the frequency to resonance and measurement of the frequency: \(0.4 \times 10^{-5}\)
2. Uncertainty of temperature of the resonator: \(0.2 \times 10^{-5}\)
3. Dimensional measurements: \(0.3 \times 10^{-5}\)
4. Residual effects of coupling holes and probes: \(0.6 \times 10^{-5}\)
5. Non-uniformity of the resonator: \(1.0 \times 10^{-5}\)
6. Uncertainty of \(Q\): \(0.5 \times 10^{-5}\)

Estimated maximum error: \(3 \times 10^{-5}\)

The uncertainties due to the first three causes can be established from the accuracy of repetition and a knowledge of the band-width of the receiver and frequency stability of the oscillator. The errors due to the other causes cannot be estimated with such certainty. That due to the coupling is estimated from the results given in §5.2. As regards the non-uniformity of the resonator it will be seen from tables 3 and 4 that the maximum deviations of the dimensions from the mean are \(+2.7\) and \(-2.0\mu\), and that the variations occur in a smooth manner. It is estimated therefore that the average dimensions are known to \(0.5\mu\). Previous results (Essen 1946) obtained with much less uniform resonators support the assumption that an average value can be taken, although it cannot be assumed that the non-uniformity of dimensions affects different modes of resonance in the same way. An overall error of \(1 \times 10^{-5}\) has therefore been allowed for these uncertainties.

The total value of the correction due to the finite resistance of the walls of the resonator is approximately \(3 \times 10^{-5}\). An uncertainty is introduced because of the imperfections in the surface of the resonator and a lack of knowledge of the precise path of the current. It is thought that these effects are to some extent taken account.
of by using the measured value of $Q$ in the calculations. The value of $Q$ is accurate to about 5%, and the overall error has been estimated at $0.5 \times 10^{-5}$.

The results obtained before and after the metrological measurements do not give quite such good agreement as was usually obtained during the preliminary measurements on dismantling and reassembling the resonator. However, in view of the importance of associating the frequency measurements closely in time with the measurement of dimensions, the values were accepted and their mean taken as giving the final result. The difference between the values of velocity obtained for the two modes is also rather larger than expected. The uncertainty in the dimensions could contribute to this, since the length and diameter affect the resonant frequencies to different extents, and the residual errors due to coupling and non-uniformity could also have different effects on the two modes.

The work described above was started in the Radio Division of the National Physical Laboratory as part of the programme of the Radio Research Board and completed in the Electricity Division as part of the research programme of the National Physical Laboratory. This paper is published by permission of the Department of Scientific and Industrial Research.

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References

Bernier, J. 1946 Onde Elect. 26, 305.
Birge, R. T. 1941 Physics, 8, 90.