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If you want a final version of a lecture, wait until after it is given.

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Chapter 1

Background Material

1.1 Introduction

This is a course on Classical Mechanics, the science of motion. In Greek philosophy, the concept of motion embraced virtually everything involving change: birth, growth, decay, displacement of objects, alterations in quality. Of these, displacement of objects, like the falling of a stone, the flight of an arrow, the sailing of a ship, came to be denoted as “local” motion. The conceptual revolution which climaxed at the turn of the 17th century, and which forms the basis of our contemporary science of mechanics, was characterized in part by the stripping away of much of the metaphorical or allegorical connotation that had been attached to motion. Once attention became focused on the description and understanding of local motion, the modern concept emerged on a time scale that was, comparatively speaking, extremely short.

1.2 Concepts and Definitions: A Vocabulary

To be useful to science, an object of investigation must be denoted unambiguously, its identification clear and universally accepted, and its characteristics quantifiable either by direct measurement or indirectly via calculation of measurable quantities. It must be possible to assign numerical values to characteristics. But measurement is a comparison against a standard unit, and so consists of both a number and a unit of measure.

The definition of a physics construct, or concept, requires, then, a procedure for measuring it, which results in a number and a unit of measure. In principle, it is also proper to include a figure of uncertainty for each measurement or calculation using data. Since every measurement is to some extent uncertain, there is no such thing as an “exact” result. The exactitude of modern mathematical sciences lies in providing honest estimates of the uncertainties associated with results.

It is a universal convention in scientific work to report, in any numerical result, all figures up to, and including, the first uncertain one. This automatically tells the reader
CHAPTER 1. BACKGROUND MATERIAL

where the uncertainty begins. Care, therefore must be taken to report results only to
known significance.

Another way to say all of this is that a name for something in physics is shorthand
for a set of operations. We’ll see shortly what this involves when we define the most ba-
sic of all physics concepts, position, instant, and mass, as well as such derived concepts
as displacement, interval, velocity, acceleration, momentum, energy, and force.

1.3 Units, Dimensions, and Physical Entities

The framework of mechanics is formed with just three independent, basic concepts:
length (denoted \( L \)), time (denoted \( T \)), and mass or mass number (denoted \( M \)). All
other mechanical concepts, such as velocity, acceleration, force, momentum, and en-
ergy, are derived from relationships between these three entities. Discussed abstractly
as length, time, and mass number, these concepts are referred to formally as dimensions.

1.3.1 Units

Physics is a quantitative, empirical science. Its language is mathematics, and it is solely
concerned with investigable and testable occurrences. Measurements, and the equa-
tions that describe, predict, or explain them, are quantitative in nature: they result in
numbers. Mathematical equations describing real physical events must have the same
dimensional quantities on each side of the equal sign. In general, it makes no sense to
add entities of different dimensions (\( L + M \), for example, is absurd), but products of
different dimensions can form interesting derived entities like velocity (\( \vec{v} = LT^{-1} \)),
force (\( \vec{F} = MLT^{-2} \)), energy (\( [E] = ML^2T^{-2} \)), and so forth. Notice that reference
to the dimension of a physical entity is indicated by enclosing the entity in brackets, [ ].

Furthermore, the numbers themselves depend on the choice of measuring tools,
which are scaled according to an arbitrary choice of units. There are several standard
and widely used systems of units. Typically, introductory physics courses in the United
States use the International System of Units (SI) or mks (meter, kilogram, second).

In this class, we will rarely deal with numbers. Not only is the choice of measure-
ment convention arbitrary and fundamentally unimportant, but learning to interpret
measurement results correctly—in terms of uncertainties and resolutions, often sub-
sumed under the heading “significant figures”—takes up more time than we can afford
and, in the meantime, leads to more confusion and silly errors in student work than
almost anything else. Besides, this subject presumably is dealt with in detail in the
laboratory. With problems enough understanding the material conceptually and math-
ematically, we’ll stick to symbolic representations of physical quantities and ignore
units.
1.4 Dimensional Analysis

Nevertheless, each term of a mathematical expression in mechanics represents a physical entity that can be understood in terms of the three basic entities. Again, the terms on each side of an equation’s equal sign must be the same product of these fundamental dimensions of $L$, $T$, and $M$. This is a minimal (necessary, but not sufficient) check that the equation is reasonable, and one should get in the habit from the start of checking that this is the case.

1.5 Scalars and Vectors

A quantity that has magnitude but no direction, like time, mass, and energy, is referred to as a scalar. In these notes, such a quantity will be denoted in italics: $t$, $m$, $E$, for example. A quantity that has a direction as well as a magnitude, such as velocity, acceleration, or force, is usually referred to as a vector.$^1$ A vector is denoted in these notes as a symbol with a small arrow above: $\vec{v}$, $\vec{a}$, $\vec{F}$, for example.

To be specified, a vector requires as many numbers, known as components, as there are dimensions, while a scalar always requires only one number.

1.6 Vectors: Graphical Representation

Any vector can be represented by a straight line. The line’s length represents the vector’s magnitude, and the angles it makes with respect to the axes of some reference frame represents the vector’s direction [see Figure 1.1].

Figure 1.1: Graphical representations, in two- and three-dimensions, of vectors $\vec{A}$ and $\vec{B}$.

In Figure 1.1-a), the magnitude and direction of vector $\vec{A}$ are indicated by the directed line from $(x_1, y_1)$ to $(x_2, y_2)$. In Figure 1.1-b), the magnitude and direction of vector $\vec{B}$ are indicated by the directed line from $(0, 0, 0)$ to $(x, y, z)$.

$^1$Note that some quantities, like rotations, have both direction and magnitude but are not vectors.
1.6.1 Graphical Addition

To add two vectors graphically (assuming they represent entities of the same dimensionality—otherwise, they can’t be added), place the tail of the second vector on the head of the first and draw a line from the tail of the first to the head of the second [see Figure 1.2]. This new line represents the magnitude and direction of the vector sum: \( \vec{E} = \vec{C} + \vec{D} \).

![Figure 1.2: Graphically adding \( \vec{C} \) and \( \vec{D} \) results in \( \vec{E} \).](image)

1.6.2 Graphical Subtraction

Graphically subtracting two vectors involves reversing the direction of the second vector and then adding as described above [see Figure 1.3].

Note that the magnitude and direction of a line define the vector completely; the placement in a reference frame, or, equivalently, the choice of coordinate system, is arbitrary. So, in these examples, \( \vec{C} \) is the same vector in Figures 1.3 and 1.2 even though its position with reference to the axes is changed.

1.7 Vectors: Component Representation

In Figure 1.1-a), the projection of \( \vec{A} \) onto the x-axis is labeled \( A_x \), and the projection of \( \vec{A} \) onto the y-axis is labeled \( A_y \). \( A_x \) and \( A_y \) are known as the components of \( \vec{A} \): \( \vec{A} = (A_x, A_y) \). The magnitude, then, of \( \vec{A} \) is

\[
| \vec{A} | = \sqrt{A_x^2 + A_y^2} \quad (1.1)
\]

and its direction is:

\[
\tan \theta = \frac{A_y}{A_x} \quad (1.2)
\]
1.7. VECTORS: COMPONENT REPRESENTATION

Figure 1.3: Graphically subtracting \( \vec{D} \) from \( \vec{C} \) results in \( \vec{E} \).

Similarly, from Figure 1.1-b), \( \vec{B} = (B_x, B_y, B_z) \), so its magnitude and direction are:

\[
| \vec{B} | = \sqrt{B_x^2 + B_y^2 + B_z^2} \quad (1.3)
\]

\[
\cos \theta = \frac{B_x}{| \vec{B} |}; \quad \tan \phi = \frac{B_y}{B_x} \quad (1.4)
\]

1.7.1 Computing Vector Components

From a vector’s magnitude and direction, one uses trigonometry to compute its components:

\[
A_x = | \vec{A} | \cos \theta; \quad A_y = | \vec{A} | \sin \theta \quad (1.5)
\]

\[
B_x = | \vec{B} | \sin \theta \cos \phi; \quad B_y = | \vec{B} | \sin \theta \sin \phi; \quad B_z = | \vec{B} | \cos \theta. \quad (1.6)
\]

1.7.2 Vector Addition with Components

Figure 1.2, again, shows \( \vec{C} + \vec{D} = \vec{E} \). \( \vec{E} \) is the vector sum of \( \vec{C} \) and \( \vec{D} \) and is referred to as the resultant. Being a vector, \( \vec{E} \) has components, \( \vec{E} = (E_x, E_y) = (C_x + D_x, C_y + D_y) \), and its magnitude and direction are \( | \vec{E} | = \sqrt{E_x^2 + E_y^2} = \sqrt{(C_x + D_x)^2 + (C_y + D_y)^2} \), and \( \tan \alpha = \frac{E_y}{E_x} = \frac{C_y + D_y}{C_x + D_x} \), respectively.

Vector addition in 3-dimensions similarly requires the sums of the three sets of components.
1.8 Vectors: Unit Vectors

Any vector $\vec{A}$ may be written as

$$\vec{A} = |\vec{A}| \hat{e}$$

where $|\vec{A}|$ is the magnitude of $\vec{A}$ and $\hat{e}$ is a unit vector in the direction of $\vec{A}$. A unit vector is dimensionless and it has, by definition, magnitude of 1, hence its name. When written in the form of Equation 1.7, the magnitude of the vector $\vec{A}$ is given by $|\vec{A}|$ and its direction by that of $\hat{e}$.

1.8.1 Unit Vectors Along Axes

It is most common to introduce unit vectors along the axes of a reference frame. For the 3-dimensional reference frame, $x$, $y$, $z$, the corresponding unit vectors are typically denoted $\hat{i}$, $\hat{j}$, $\hat{k}$. And, since every vector is the resultant of its components,

$$\vec{B} = (B_x, B_y, B_z) = B_x \hat{i} + B_y \hat{j} + B_z \hat{k}.$$  

If the vector of Figure 1.1-b) were to represent a position, about which we will speak in the next chapter, then it represents a special case of vector known as a radius vector $\vec{r}$ which is a directed line segment from the origin $O(0, 0, 0)$ to a point $P(x, y, z)$. We then write

$$\vec{r} = x \hat{i} + y \hat{j} + z \hat{k}$$

$$|\vec{r}| = \sqrt{x^2 + y^2 + z^2}$$

$$\cos \theta = \frac{z}{|\vec{r}|}; \quad \tan \phi = \frac{y}{x}$$

For motion in 1-dimension, we use the symbol $s$ for position; in 2- or 3-dimensions, we use $\vec{r}$.

1.9 Vectors: Multiplication

In this course, we employ three types of multiplication with vectors.

1.9.1 Multiplication of a Vector by a Scalar

Given a scalar $\lambda$, the quantity $\lambda \vec{A}$ is a vector with a magnitude $|\lambda| \ |\vec{A}|$ (the absolute value of $\lambda$ times the magnitude of $\vec{A}$), and a direction the same as $\vec{A}$ if $\lambda > 0$ or the reverse of $\vec{A}$ if $\lambda < 0$. Of course, if $\lambda = 0$, the $\lambda \vec{A} = \vec{0}$, the null vector, a vector with no magnitude or direction.
1.9. VECTORS: MULTIPLICATION

1.9.2 Scalar or Dot Product of Two Vectors

Given two vectors, $\vec{A}$ and $\vec{B}$, whose directions differ by an angle $\theta$, their dot product, written $\vec{A} \cdot \vec{B}$ results in a scalar quantity defined

$$\vec{A} \cdot \vec{B} \equiv |\vec{A}| |\vec{B}| \cos \theta, \quad (1.12)$$

where, again, $\theta$ is the angle between $\vec{A}$ and $\vec{B}$, the included angle. Note, that the scalar result is maximally positive when $\vec{A}$ and $\vec{B}$ are parallel, maximally negative when they are anti-parallel, and zero when they are perpendicular. Note also that the dot product is commutative: $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A}$.

**Dot Product of Rectangular Unit Vectors**

Because $\hat{i}$, $\hat{j}$, and $\hat{k}$ are mutually perpendicular and of magnitude 1, dot products between and among them are:

$$\hat{i} \cdot \hat{i} = \hat{j} \cdot \hat{j} = \hat{k} \cdot \hat{k} = 1 \quad (1.13)$$

$$\hat{i} \cdot \hat{j} = \hat{j} \cdot \hat{k} = \hat{k} \cdot \hat{i} = 0 \quad (1.14)$$

**Dot Product in Component Form**

Consider 2-dimensional vectors $\vec{A} = A_x \hat{i} + A_y \hat{j}$ and $\vec{B} = B_x \hat{i} + B_y \hat{j}$. Then

$$\vec{A} \cdot \vec{B} = (A_x \hat{i} + A_y \hat{j}) \cdot (B_x \hat{i} + B_y \hat{j}).$$

Assuming the distributive law holds,

$$\vec{A} \cdot \vec{B} = A_x B_x \hat{i} \cdot \hat{i} + A_x \hat{i} \cdot B_y \hat{j} + A_y \hat{j} \cdot B_x \hat{i} + A_y B_y \hat{j} \cdot \hat{j}$$

$$= A_x B_x (\hat{i} \cdot \hat{i}) + A_x B_y (\hat{i} \cdot \hat{j}) + A_y B_x (\hat{j} \cdot \hat{i}) + A_y B_y (\hat{j} \cdot \hat{j})$$

$$= A_x B_x + A_y B_y,$$

where Equations 1.13 and 1.14 have been used.

Recall, from Equation 1.2, that the directional angle of a vector (say, here, $\vec{A}$) is given by $\tan \theta_A = A_y / A_x$. But $\tan \theta_A = \sin \theta_A / \cos \theta_A$, so if we divide top and bottom of the ratio $A_y / A_x$ by $|\vec{A}|$, we get

$$\tan \theta_A = \frac{\sin \theta_A}{\cos \theta_A} = \frac{A_y / |\vec{A}|}{A_x / |\vec{A}|}.$$

Thus, $\sin \theta_A = A_y / |\vec{A}|$ and $\cos \theta_A = A_x / |\vec{A}|$. And similarly for $\vec{B}$. Then,

$$\vec{A} \cdot \vec{B} = A_x B_x + A_y B_y$$
\[ \begin{align*}
\vec{A} \cdot \vec{B} &= A_x B_x + A_y B_y \\
&= |\vec{A}| |\vec{B}| \cos \theta,
\end{align*} \]

Hence, we arrive at the definition of the dot product:

Now, \( \theta_A - \theta_B \) is the magnitude of the angle between \( \vec{A} \) and \( \vec{B} \). We’ll call it \( \theta \). This notion generalizes to more than two dimensions. In 3-dimensions, we have

\[ \vec{A} \cdot \vec{B} = A_x B_x + A_y B_y + A_z B_z = |\vec{A}| |\vec{B}| \cos \beta, \]

where \( \beta \) is the angle between \( \vec{A} \) and \( \vec{B} \) in 3-dimensions.

**A Geometric Interpretation of the Dot Product**

The dot product may be interpreted as the product of the magnitude of one vector with the projection of a second vector onto the first [see Figure 1.4]. To find the projection of one vector onto another, place the vectors tail-to-tail and drop a perpendicular from the head of one to the body of the other, forming a right triangle. If \( \theta > \frac{\pi}{2} \), the projection is negative. Trigonometry gives that [see Figure 1.4-a)] the projection of \( \vec{A} \) onto \( \vec{B} \) is \( |\vec{A}| \cos \theta \), and that [see Figure 1.4-b)] the projection of \( \vec{B} \) onto \( \vec{A} \) is \( |\vec{B}| \cos \theta \).

**1.9.3 Vector or Cross Product of Two Vectors**

Given two vectors, \( \vec{A} \) and \( \vec{B} \), whose directions differ by an angle \( \theta \), their cross product, written \( \vec{A} \times \vec{B} \), results in a vector quantity with magnitude

\[ |\vec{A} \times \vec{B}| = |\vec{A}| |\vec{B}| \sin \theta, \quad (1.15) \]

where, again, \( \theta \) is the angle between \( \vec{A} \) and \( \vec{B} \), the included angle. Note, that this magnitude is a maximum when \( \vec{A} \) and \( \vec{B} \) are perpendicular, and zero when they are parallel or anti-parallel. The direction of this product is defined by convention to be perpendicular to both \( \vec{A} \) and \( \vec{B} \) in the sense of a so-called right-hand rule: beginning with the fingers of your right hand pointing along the direction of \( \vec{A} \), curl your fingers toward \( \vec{B} \) and extend your thumb; the thumb points in the direction of the cross product [see Figure 1.5].

Note that as a result of this direction convention, the cross product is anti-commutative: \( \vec{A} \times \vec{B} = - (\vec{B} \times \vec{A}) \).
Cross Product of Rectangular Unit Vectors

Because \( \hat{i}, \hat{j}, \) and \( \hat{k} \) are mutually perpendicular and of magnitude 1, cross products between and among them are:

\[
\begin{align*}
\hat{i} \times \hat{i} &= \hat{j} \times \hat{j} = \hat{k} \times \hat{k} = \vec{0} \\
\hat{i} \times \hat{j} &= \hat{k}; & \hat{j} \times \hat{k} &= \hat{i}; & \hat{k} \times \hat{i} &= \hat{j} \\
\hat{j} \times \hat{i} &= -\hat{k}; & \hat{k} \times \hat{j} &= -\hat{i}; & \hat{i} \times \hat{k} &= -\hat{j}
\end{align*}
\]

(1.16) (1.17) (1.18)

Cross Product in Component Form

Consider 3-dimensional vectors \( \vec{A} = A_x \hat{i} + A_y \hat{j} + A_z \hat{k} \) and \( \vec{B} = B_x \hat{i} + B_y \hat{j} + B_z \hat{k} \). Then

\[
\vec{A} \times \vec{B} = (A_x \hat{i} + A_y \hat{j} + A_z \hat{k}) \times (B_x \hat{i} + B_y \hat{j} + B_z \hat{k}).
\]

Assuming the distributive law holds,
\[ \vec{A} \times \vec{B} = A_x \hat{i} \times B_x \hat{i} + A_x \hat{i} \times B_y \hat{j} + A_x \hat{i} \times B_z \hat{k} + \\
A_y \hat{j} \times B_x \hat{i} + A_y \hat{j} \times B_y \hat{j} + A_y \hat{j} \times B_z \hat{k} + \\
A_z \hat{k} \times B_x \hat{i} + A_z \hat{k} \times B_y \hat{j} + A_z \hat{k} \times B_z \hat{k} \]
\[
= A_x B_x (i \times i) + A_x B_y (i \times j) + A_x B_z (i \times k) + \\
A_y B_x (j \times i) + A_y B_y (j \times j) + A_y B_z (j \times k) + \\
A_z B_x (k \times i) + A_z B_y (k \times j) + A_z B_z (k \times k) \]
\[
= A_x B_y \hat{k} + A_x B_z (-\hat{j}) + A_y B_x (-\hat{k}) + A_y B_z \hat{i} + A_z B_x \hat{j} + A_z B_y (-\hat{i}) \]
\[
= (A_y B_z - A_z B_y) \hat{i} + (A_z B_x - A_x B_z) \hat{j} + (A_x B_y - A_y B_x) \hat{k} \]

where Equations 1.16, 1.17 and 1.18 have been used. Thus, if \( \vec{C} = (C_x, C_y, C_z) = \vec{A} \times \vec{B} \), then \( C_x = A_y B_z - A_z B_y, C_y = A_z B_x - A_x B_z, \) and \( C_z = A_x B_y - A_y B_x \).

An equivalent way to express \( \vec{A} \times \vec{B} \) is as a determinant expanded with respect to the first row:

\[
\begin{vmatrix}
    \hat{i} & \hat{j} & \hat{k} \\
    A_x & A_y & A_z \\
    B_x & B_y & B_z \\
\end{vmatrix}
\]

A Geometric Interpretation of the Cross Product

The cross product of two vectors may be thought of as the product of the magnitude of one vector and the perpendicular component of the other: \( |\vec{A}| \) times the perpendicular component of \( |\vec{B}| \), or, equivalently, \( |\vec{B}| \) times the perpendicular component of \( |\vec{A}| \) [see Figure 1.6], which we may write as

\[
|\vec{A} \times \vec{B}| = A_\perp |\vec{B}| = |\vec{A}| B_\perp.
\]

![Figure 1.6](image-url)

Figure 1.6: a) \( |\vec{A} \times \vec{B}| = A_\perp |\vec{B}| \); b) \( |\vec{A} \times \vec{B}| = |\vec{A}| B_\perp \).
Chapter 2

Rectilinear Motion

2.1 Position and Displacement

To create a numerical description of an object’s position along a straight line, we set up a series of numerical markers [see Figure 2.1]. Denoting an arbitrarily chosen “origin” point by 0, we mark other points +1, +2, -1, -2, etc., laying off a conveniently oriented coordinate scale (number line). The direction we call positive (or negative) is up to us. The size of the spacing between integer numbers along the line being arbitrary, as well, we are free to adopt any unit of length we wish, although, for clarity and ease of communication, a national or international standard is probably a good choice.

Such a reference line or coordinate axis is an example of a reference frame or frame of reference. To describe positions in two-(three-)dimensional space, we erect two (three) such axes, usually perpendicular to each other, to obtain a two-(three-)dimensional reference frame.

In 1-dimension, we use the symbol $s$ to denote any individual position, and symbols with subscripts ($s_0$, $s_1$, $s_2$, etc.) to denote particular positions we wish to distinguish from each other. These symbols represent only positions relative to an arbitrarily chosen origin, not distances traversed by the body.

Let us denote a first position by $s_1$ and a second position by $s_2$. The number $s_2 - s_1$ gives us information about what we might call change of position or displacement.
We will use the symbol $\Delta s$ as a shorthand for the number $s_2 - s_1$. That is,

$$\Delta s \equiv s_2 - s_1,$$

(2.1)

where $\Delta s$, again, is called **displacement** or **change of position**.

$\Delta s$ has the following algebraic properties: it may be positive, negative, or zero depending on the numerical values and algebraic signs of $s_2$ and $s_1$. The resulting sign indicates the direction of the displacement (in terms an arbitrarily chosen orientation) as long as the positions $s_1$ and $s_2$ are time-ordered (that is, the object occupied $s_1$ before it occupied $s_2$).

### 2.2 Instant and Interval of Time

**Instants** are assigned numbers in much the same way as position. In fact, the former depends on the latter being well-established. Fortunately, though, there are fewer conventional time units than space units. Instants are typically measured by observing the position of some object which, in effect, counts a regular variation or oscillation. The position of the hand of a clock, for example, counts oscillations of a balance wheel or a pendulum or a vibrating quartz crystal. By associating such positions with clock readings or **instants of time**, a sequence of instants can be matched to a number line [see Figure 2.2]. We use the symbols $t$ to indicate a general instant of time, and $t_0$, $t_1$, $t_2$, etc., for specific clock readings.

![Figure 2.2: Instants of time $t$ represented on a number line.](attachment:figure2.2.png)

We bring an intuitive conception of the natural “flow” of time to this representation and adopt the convention of labeling succeeding instants with increasingly positive numbers. In other words, time progresses only in the positive “direction.” The instant we denote $t = 0$, just like the position we denote $s = 0$, is arbitrary. The negative numbers here, though, contain no implication whatsoever of time running backward, or negative time intervals. Negative $t$ simply refers to an instant which precedes the arbitrarily (though conveniently) chosen $t = 0$.

To calculate an **interval of time**, we must take a difference:

$$\Delta t \equiv t_2 - t_1,$$

(2.2)

where $t_2$ is always taken as the instant later in the sequence than $t_1$. Thus, we define time interval $\Delta t$ as an intrinsically positive quantity.
2.3  \( s - t \) Histories

Say we observe an object is located at some position \( s_1 \) when the hand of the clock is at “position” or instant \( t_1 \). Thus we observe the simultaneous occurrence of a pair of events—position of the object and position of the hand of the clock. Similarly, we observe the coincidence of a second pair of events which we associate with the symbols \( t_2 \) and \( s_2 \). Now, having the pairs of numbers or coordinates \((t_1, s_1)\) and \((t_2, s_2)\). These, and other pairs, can then be plotted on a position versus instant graph or position versus clock reading graph to give an \( s - t \) history of the object.

We set up a pair of coordinate axes [see Figure 2.3], and, instead of labeling them \( y \) and \( x \) as in elementary algebra, we label them \( s \) and \( t \). In this representation, each “event” in the rectilinear (or straight-line) motion of a particle (that is, its position \( s \) occupied at instant \( t \)) becomes associated with a point in the \( s - t \) plane, defined by the two coordinate lines (axes). An accumulation of data—corresponding values of \( t \) and \( s \)—can then be “plotted” or “graphed” to give a visual picture of the history of an observed motion; or, conversely, a line or curve drawn on an \( s - t \) diagram can be interpreted as the history of a hypothetical motion. In coordinate representations of this kind, the horizontal coordinate, in this case \( t \), is called the abscissa, and the vertical is called the ordinate.

Figure 2.3: A sequence of positions \( s \) occupied at successive instants of time \( t \).

We may also calculate the displacement \( \Delta s \) and the corresponding time interval \( \Delta t \) from pairs of such pairs.
2.4 Velocity

2.4.1 Average Velocity

We signify the ratio of displacement to its corresponding time interval the average velocity:

\[ \bar{v} \equiv \frac{\Delta s}{\Delta t} \quad (2.3) \]

Equation 2.3 can be rearranged: \( \Delta s = \bar{v} \Delta t \) or \( s_2 - s_1 = \bar{v}(t_2 - t_1) \). If we indicate initial values with the subscript \( i \) (which, be aware, does not necessarily imply that \( t_i = 0 \)) and any final values without subscripts, the last equation becomes \( s - s_i = \bar{v}(t - t_i) \), or

\[ s = s_i + \bar{v}(t - t_i) \quad (2.4) \]

That is, the position \( s \) at some instant \( t \) is numerically equal to the position at the initial instant \( s_i \) plus the average velocity \( \bar{v} \) times the time interval since the initial instant \( t - t_i = \Delta t \)

Note that the average velocity \( \bar{v} \) has algebraic characteristics that derive from those of \( \Delta s \) and \( \Delta t \). The algebraic sign of \( \Delta s \), recall, indicates the direction (relative to the chosen reference frame) of displacement. \( \Delta t \) is positive definite. Thus, the sign of the average velocity indicates its direction. The dimension of velocity is \( LT^{-1} \).

2.4.2 Instantaneous Rectilinear Velocity

In more than one dimension, the vectorial characteristics of displacement become evident. The direction therefore is denoted with appropriate angles or unit vectors rather than simply an algebraic sign. In 3-dimensions, we write:

\[ \vec{r} = (x, y, z) = x\hat{i} + y\hat{j} + z\hat{k}. \quad (2.5) \]

If we consider two positions relative to our chosen origin, \( \vec{r}_1 = (x_1, y_1, z_1) \) and \( \vec{r}_2 = (x_2, y_2, z_2) \), then the vector displacement in 3-dimensions becomes

\[ \Delta \vec{r} = \vec{r}_2 - \vec{r}_1 = (x_2 - x_1, y_2 - y_1, z_2 - z_1) = (\Delta x, \Delta y, \Delta z). \quad (2.6) \]

\( \Delta t \) of course is a scalar, and so, therefore, is \( \frac{\Delta \vec{r}}{\Delta t} \). Then the ratio \( \frac{\Delta \vec{r}}{\Delta t} \) is an example of multiplication of a vector, \( \Delta \vec{r} \) by a scalar, \( \frac{1}{\Delta t} \). We may then take the limit as \( \Delta t \) becomes infinitesimally small to define instantaneous rectilinear velocity:

\[ \vec{v} \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{r}}{\Delta t} \equiv \frac{d\vec{r}}{dt} \equiv \dot{\vec{r}}; \quad (2.7) \]

where a single dot over a variable indicates differentiation with respect to time.

In terms of rectangular unit vectors,
2.5. ACCELERATION

\[ \vec{v} = v_x \hat{i} + v_y \hat{j} + v_z \hat{k} \]

\[ = \frac{dx}{dt} \hat{i} + \frac{dy}{dt} \hat{j} + \frac{dz}{dt} \hat{k} \]

\[ \equiv \vec{r} = \dot{x} \hat{i} + \dot{y} \hat{j} + \dot{z} \hat{k} . \]  

(2.8)

Notice that the instantaneous velocity is the **derivative** of the position versus time curve. That is, if one plots position as a function of time, the tangent of the resulting curve at a given instant is the instantaneous rectilinear velocity at that instant, both its magnitude and the direction.

2.5 Acceleration

An object has instantaneous velocity \( \vec{v}_1 \) at \( t_1 \), and instantaneous velocity \( \vec{v}_2 \) at \( t_2 \), and so forth. Such pairs \((t_1, v_1), (t_2, v_2), \ldots\) may be plotted to graphically present a velocity versus time history.

If \( \vec{v}_i = \vec{v}_j \) regardless of the instants at which the velocity is measured, then we say that the motion is **constant** or **uniform**. The velocity versus time graph will be a horizontal straight line. The graph of position versus time in this case will be a straight line (since the tangent, or slope, of the resulting curve is constant). The straight line will be horizontal (parallel to the clock time axis) if this constant velocity is \( \vec{0} \) (the position doesn't change with time); it will have positive pitch if the velocity is in the positive direction and negative pitch if the velocity is in the negative direction.

In general, though, the **change in velocity**, \( \Delta \vec{v} \equiv \vec{v}_2 - \vec{v}_1 \), will not be zero. That is, for general motion, the magnitude or the direction of the velocity—or both—may change. Velocity change involves vector subtraction and results in a zero vector only if the two vectors are identical.

2.5.1 Instantaneous Rectilinear Acceleration

The rate at which the velocity changes, during a finite interval of time, is called the **instantaneous rectilinear acceleration**:

\[ \ddot{\vec{v}} \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{v}}{\Delta t} \equiv \frac{d\vec{v}}{dt} = \frac{dv_x}{dt} \hat{i} + \frac{dv_y}{dt} \hat{j} + \frac{dv_z}{dt} \hat{k} \]

\[ \equiv \dot{\vec{v}} = \dot{x} \hat{i} + \dot{y} \hat{j} + \dot{z} \hat{k} = \vec{a} \equiv \frac{d^2 \vec{r}}{dt^2} \]

\[ = \frac{d^2 x}{dt^2} \hat{i} + \frac{d^2 y}{dt^2} \hat{j} + \frac{d^2 z}{dt^2} \hat{k} \]

\[ \equiv \ddot{\vec{r}} = \ddot{x} \hat{i} + \ddot{y} \hat{j} + \ddot{z} \hat{k} = a_x \hat{i} + a_y \hat{j} + a_z \hat{k} , \]

(2.9)

where double dots over a variable indicate a second derivative with respect to time. Notice that the instantaneous acceleration is the derivative of the velocity versus time curve. That is, if one plots velocity as a function of time, the tangent of the resulting
curve at a particular instant is the instantaneous rectilinear acceleration at that instant, giving both the magnitude and the direction.

2.5.2 Uniformly Accelerated Rectilinear Motion

If \( \vec{a} \) remains constant in both magnitude and direction, the motion is called uniform acceleration in rectilinear motion. The plot of velocity versus time is a straight line, the slope of which gives the magnitude and direction of the constant acceleration. These remain the same over all time intervals, \( \Delta t \), and not just for an infinitesimal interval \( dt \). In such a case, it is most convenient to set an axis of your reference frame (or coordinate system) along the line of the acceleration. The positive direction may point either parallel or anti-parallel to \( \vec{a} \). Having done so, the vector arithmetic reduces to working with signed numbers, where the sign indicates the direction: positive means parallel to the axis and negative means anti-parallel. We can drop the vector notation, but we must then be careful with (especially when interpreting) the sign. So, \( \vec{a} \) becomes \( \pm a \), \( \vec{v} \) becomes \( \pm v \), and \( \vec{r} \) becomes \( \pm s \), where the sign depends on the direction relative to the chosen positive direction of the axis in the case of \( a \) and \( v \) and the location relative to the arbitrarily chosen origin in the case of \( s \). Furthermore, with \( \vec{a} = a \) a constant, then

\[
a = \frac{\Delta v_\parallel}{\Delta t},
\]

where \( v_\parallel \) is the projection of \( \vec{v} \) onto the axis along the direction of the acceleration. For 1-dimensional motion, \( v_\parallel = v \), which is the signed velocity we can use in place of vector notation (for this special case). Thus, for 1-dimensional, uniformly accelerated motion, we have \( a = \frac{\Delta v}{\Delta t} \). Let us, as previously, identify initial variables with the subscript \( i \) and the final variables without a subscript. So, again for 1-dimensional, uniformly accelerated motion only,

\[
v = v_i + a(t - t_i),
\]

for the special case of 1-dimensional, uniformly accelerated motion only. This, you’ll recall, is similar in form to Equation 2.4, \( s = s_i + \frac{1}{2}(v + v_i)(t - t_i) \).

Now, consider a velocity versus clock time plot for 1-dimensional, uniformly accelerated motion [see Figure 2.4]. In particular, consider the area under the curve (here, the “curve” is a straight line, because the acceleration is assumed constant) between \( t = t_i \), when \( v = v_i \), and clock reading \( t \), when the velocity is \( v \). The shape under consideration is a right trapezoid composed of a rectangle of base \( \Delta t = t - t_i \) and height \( v_i \), and a triangle of base \( \Delta t = t - t_i \) and height \( \Delta v = v - v_i \).

The area, then, is \( v_i(t - t_i) + \frac{1}{2}(v - v_i)(t - t_i) = \frac{1}{2}(v + v_i)(t - t_i) \). Notice that the area is a product of a sum of velocities (which results in a quantity of the same dimension as velocity, \( LT^{-1} \), and a time interval, of dimension \( T \). The product therefore has dimension \( L \).

In Equation 2.4, we also have a product of a velocity and a time interval, the result of which is a displacement. We might surmise that \( \frac{1}{2}(v + v_i)(t - t_i) \) is a displacement,
2.5. ACCELERATION

under the condition that the motion is linear and uniformly accelerated. If our guess is correct, then we have to identify

$$v = \frac{v + v_i}{2},$$  \hspace{1cm} (2.12)

for uniformly accelerated, rectilinear motion.\(^1\) Equation 2.4 now, under these restrictions, takes the form $s = s_i + \frac{1}{2}(v + v_i)(t - t_i)$. But we know from Equation 2.9 that $v = v_i + a(t - t_i)$. Substituting this expression for the velocity at clock reading $t$ into the revised Equation 2.4 (both of which hold only if the motion is uniformly accelerated rectilinear motion) we get, after some algebra:

$$s = s_i + v_i(t - t_i) + \frac{1}{2}a(t - t_i)^2. \hspace{1cm} (2.13)$$

Solving Equation 2.11 for $\Delta t$ and substituting into Equation 2.13, we arrive at the third important equation describing uniformly accelerated rectilinear motion:

$$v^2 = v_i^2 + 2a(s - s_i). \hspace{1cm} (2.14)$$

So, as a result of the definition of acceleration and by reasoning that, for uniformly accelerated rectilinear motion, $\bar{v} = \frac{v_i + v}{2}$, we have three equations, each describing such motion in terms of three of the four variables of interest: displacement, velocity, acceleration, and time interval. These equations, again, are:

\(^1\) Be sure to understand that this identification of average velocity with the arithmetic average of initial and final velocities holds only in the case of uniformly accelerated, rectilinear motion; it is not a general relationship.
Equation 2.11: \( v = v_i + a \Delta t \)
Equation 2.13: \( s = s_i + v_i \Delta t + \frac{1}{2} a (\Delta t)^2 \)
Equation 2.14: \( v^2 = v_i^2 + 2a s \).

Gravitational Acceleration

Because every object falling freely near the surface of the earth accelerates at an almost constant rate toward the center of the earth, such motion has been used to study the relations just given. To a very good approximation, the three equations hold, justifying our guess that, for this sort of motion, the average velocity is numerically equal to the arithmetic average of the initial and final velocities. In fact, this relationship can be derived using the integral methods for finding averages:

\[
\bar{v} \equiv \frac{1}{\Delta v} \int_{v_i}^{v} v \, dv = \frac{1}{2} \frac{v^2 - v_i^2}{v - v_i} = \frac{v + v_i}{2}.
\]

The near constancy of free fall acceleration, and the importance of gravitational acceleration to all aspects of terrestrial mechanics, has earned this acceleration its own symbol, \( g \), which we will employ frequently.

2.5.3 An Example Problem

A stone is thrown vertically upward with an initial velocity \( v_0 \) from the edge of a cliff height \( h \) above the value below. The stone rises and falls along a straight line just beyond the edge of the cliff [see Figure 2.5].

We assume that the motion is uniformly accelerated and rectilinear, so that we can use the kinematic equations 2.11, 2.13, and 2.14.

The following [see Figure 2.6] are sketches of acceleration versus clock time, velocity versus clock time, and position versus clock time. In the plots, we have taken \( t_i = 0 \) and \( s_i = 0 \). Notice that, having assigned upward the positive direction, the velocity always decreases with time. The dashed lines highlight key instants, but have no physical significance.

1. How high does the stone rise?

At the top of its trajectory, the stone’s velocity is instantaneously 0. This is not to say that it stops, as it’s acceleration remains uniform: from one instant to the next the stone’s velocity is changing. You might think of the maximum height as a transition point between positive and negative velocity. Then, using Equation 2.14 \( v^2 = v_i^2 + 2a(s - s_i) \), we have
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Figure 2.5: Stone’s trajectory. Path overlaps, but is offset here for clarity.

\[ 0 = v_0^2 - 2g(s_{\text{max}} - 0) \]
\[ s_{\text{max}} = \frac{v_0^2}{2g} \]

2. At what instant does it arrive at its highest point?

Using Equation 2.11 \( v = v_i + a\Delta t = v_i + a(t - t_i) \), we have

\[ 0 = v_0 - gt_{\text{max}} \]
\[ t_{\text{max}} = \frac{v_0}{g} \]

3. What is the stone’s velocity when it returns past the edge of the cliff?

Using Equation 2.14 \( v^2 = v_i^2 + 2a(s - s_i) \), we have

\[ v^2 = v_0^2 - 2g(0 - 0) \]
\[ v^2 = v_0^2 \]
\[ v = \pm v_0 \]

There are two values for the stone’s velocity at the cliff’s edge: \(+v_0\) is the stone’s initial velocity when it begins its ascension; so, \(-v_0\) must be its velocity on its return to position \( s = 0 \). The stone’s velocity is equal in magnitude but opposite in direction when it returns to the place it was thrown. You should be able to prove that this relationship holds for any position above the cliff edge.
4. At what instant does the stone strike the ground at the foot of the cliff?

Asked about a relationship between position and time, knowing the constant acceleration and the initial velocity and position as well as the final position $s = -h$, we use Equation 2.13 $s = s_i + v_i \Delta t + \frac{1}{2}a(\Delta t)^2$:

$$s = -h = 0 + v_0(t - 0) - \frac{1}{2}g(t - 0)^2$$

$$-h = v_0t - \frac{1}{2}gt^2$$

This is a quadratic equation which we must solve for $t$. It will yield two solutions, from which we must decide the correct one.

$$\frac{1}{2}gt^2 - v_0t - h = 0$$

$$t = \frac{v_0 \pm \sqrt{v_0^2 - 4 \left( \frac{1}{2}g \right) (-h)}}{2 \left( \frac{1}{2}g \right)}$$

$$= \frac{v_0 \pm \sqrt{v_0^2 + 2gh}}{g}$$

Now the argument of the square-root is positive definite (all quantities are positive and added together) and larger than $v_0^2$, so that the square-root will be larger than $v_0$. Therefore, the +-sign will yield a positive clock time while the +-sign yields a negative clock time. Given the statement of the problem, a negative clock time is not physically possible, but for a point of information the negative clock time represents the instant the stone would have passed $-h$ on the way up to the cliff-top so as to have an upward velocity of $v_0$ and a position $0$ at clock time $t = 0$. In any case, the clock time when the stone hits the ground is

$$t = v_0 + \sqrt{v_0^2 + 2gh}$$

if the clock time was 0 when the stone was first thrown. At clock time $t > \frac{v_0 + \sqrt{v_0^2 + 2gh}}{g}$ the stone obviously is lying on the ground.
2.5. ACCELERATION

\[ \Delta v = v_f - v_0 - g \Delta t \]

\[ s_{\text{max}} = v_0 \Delta t - \frac{1}{2} g (\Delta t)^2 \]

Figure 2.6: \( a \) vs. \( t \), \( v \) vs. \( t \), and \( s \) vs. \( t \) plots.
Chapter 3

Planar Motion with Constant Acceleration

We noted previously the general representations of position, velocity, and acceleration in three dimensions:

Equation 2.5: \[ \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} \]
Equation 2.8: \[ \vec{v} = v_x\hat{i} + v_y\hat{j} + v_z\hat{k} = \dot{x}\hat{i} + \dot{y}\hat{j} + \dot{z}\hat{k} \]
Equation 2.9: \[ \vec{a} = a_x\hat{i} + a_y\hat{j} + a_z\hat{k} = \ddot{x}\hat{i} + \ddot{y}\hat{j} + \ddot{z}\hat{k}. \]

For motion in a plane, i.e., 2-dimensional motion, these equations reduce to

\[ \vec{r} = x\hat{i} + y\hat{j} \]  
\[ \vec{v} = v_x\hat{i} + v_y\hat{j} = \dot{x}\hat{i} + \dot{y}\hat{j} \]  
\[ \vec{a} = a_x\hat{i} + a_y\hat{j} = \ddot{x}\hat{i} + \ddot{y}\hat{j}. \]  

If the acceleration is constant, then \( a_x \) (\( \ddot{x} \)) and \( a_y \) (\( \ddot{y} \)) are both constant.

The magnitudes and directions of these vectors are:

\[ |\vec{r}| = (x^2 + y^2)^{1/2} \quad \text{tan } \gamma = \frac{y}{x} \]  
\[ |\vec{v}| = (\dot{x}^2 + \dot{y}^2)^{1/2} \quad \text{tan } \beta = \frac{\dot{y}}{\dot{x}} \]  
\[ |\vec{a}| = (\ddot{x}^2 + \ddot{y}^2)^{1/2} \quad \text{tan } \alpha = \frac{\ddot{y}}{\ddot{x}}, \]

where \( \gamma \), \( \beta \), and \( \alpha \) are the angles between \( \vec{r} \) and the \( x \)-axis, \( \vec{v} \) and the \( x \)-axis, and \( \vec{a} \) and the \( x \)-axis, respectively.

\( \vec{r} \), \( \vec{v} \), and \( \vec{a} \) are related to one another, in the case of constant acceleration, by relatively simple integration with respect to time:
\[ \vec{v} = \int_{t_i}^{t} \vec{a} dt = \vec{v}_i + \vec{a} \Delta t \quad (3.7) \]

\[ \vec{r} = \int_{t_i}^{t} \vec{v} dt = \vec{r}_i + \vec{v}_i \Delta t + \frac{1}{2} \vec{a} (\Delta t)^2, \quad (3.8) \]

where \( \vec{v}_i \) and \( \vec{r}_i \) are the initial values (the values at \( t = t_i \)) of \( \vec{v} \) and \( \vec{r} \), respectively, and \( \Delta t = t - t_i \).

Under the condition of constant acceleration, we can add

\[ |\vec{v}|^2 = |\vec{v}_i|^2 + 2 \vec{a} \cdot \Delta \vec{r}. \quad (3.9) \]

That is, the squared velocity changes as twice the dot product of the acceleration and displacement, or, more concretely, as twice the product of the magnitude of the displacement and the parallel component of the acceleration.

The components, then, of Equations 3.7, 3.8, and 3.9 lead for each coordinate to relations exactly analogous to those we’ve seen before in 1-dimensional motion:

\[ \ddot{x} = \frac{1}{2} (\dot{x}_i + \dot{x}) \quad \ddot{y} = \frac{1}{2} (\dot{y}_i + \dot{y}) \quad (3.10) \]

\[ \dot{x} = \dot{x}_i + \ddot{x} \Delta t \quad \dot{y} = \dot{y}_i + \ddot{y} \Delta t \quad (3.11) \]

\[ x = x_i + \dot{x}_i \Delta t + \frac{1}{2} \ddot{x} (\Delta t)^2 \quad y = y_i + \dot{y}_i \Delta t + \frac{1}{2} \ddot{y} (\Delta t)^2 \quad (3.12) \]

\[ \dot{x}^2 = \dot{x}_i^2 + 2 \ddot{x} \Delta x \quad \dot{y}^2 = \dot{y}_i^2 + 2 \ddot{y} \Delta y, \quad (3.13) \]

where \( \Delta x = x - x_i \) and \( \Delta y = y - y_i \) are the displacements in the \( x \)- and \( y \)-directions, respectively.

It is important to understand that to assume uniformity (constancy) of acceleration, in both magnitude and direction, is not to imply that the motion is necessarily rectilinear (in a straight line). In fact, the general motion in a plane will be parabolic, although the degenerate case, when one of the initial velocities is and remains zero, is rectilinear.

To see this, first recall that we are free to set up our axis in any orientation we choose. The choose, for example, the \( x \)-axis to be parallel to acceleration. Thus, \( \vec{a} = a_x \hat{i} = \ddot{x} \hat{i} \), and \( \vec{y} = 0 \). Then, Equations 3.11 become:

\[ x = x_i + \dot{x}_i \Delta t + \frac{1}{2} \ddot{x} (\Delta t)^2 \quad y = y_i + \dot{y}_i \Delta t. \]

Solving the second of these equations for \( \Delta t \) and substituting the result for \( \Delta t \) in the first, we get

\[ x = x_i + \frac{\dot{x}_i}{y_i} (y - y_i) + \frac{\ddot{x}}{2y_i^2} (y - y_i)^2, \quad \text{or} \]

\[ (y - y_i + \frac{\dot{x}_i y_i}{\dot{x}})^2 = 2 \frac{y_i^2}{\dot{x}} (x - x_i + \frac{\dot{x}_i^2}{2\ddot{x}}), \]
CHAPTER 3. PLANAR MOTION WITH CONSTANT ACCELERATION

Two particularly interesting numbers describing the flight of a projectile on earth are the time of flight and its horizontal range. Consider the case in which a projectile is launched with velocity $|\vec{v}| = v_i$ at angle $\alpha$ relative to a horizontal terrain so that it starts at level $y_i$ and returns to level $y = y_i$. We find the time of flight with Equation 3.11

$$y = y_i + v_i \sin \alpha \Delta t - \frac{1}{2} g (\Delta t)^2,$$

where $\dot{y} = v_i \sin \alpha$, and $\ddot{y} = -g$:

$$y = y_i + v_i \sin \alpha \Delta t - \frac{1}{2} g (\Delta t)^2.$$

This equation has two solutions: $\Delta t = 0$, which corresponds to the start of the motion at $(x_i, y_i)$, and a second value of $\Delta t$, which we may denote $\Delta t_R$, given by

$$\Delta t_R = \frac{2v_i \sin \alpha}{g},$$

which is called the time of flight. The horizontal displacement or flight range $R$ is obtained by substituting the time of flight $\Delta t_R$ for $\Delta t$ in the horizontal version of Equation 3.11 $x = x_i + \dot{x}_i \Delta t + \frac{1}{2} \ddot{x}(\Delta t)^2$, where, now, $\dot{x}_i = v_i \cos \alpha$ and $\ddot{x} = 0$.

$$R = \Delta x = v_i \cos \alpha \frac{2v_i \sin \alpha}{g} = \frac{v_i^2 \sin 2\alpha}{g}.$$

This equation informs us how the range of a projectile over horizontal terrain must depend on the magnitude and elevation of its initial velocity. If the initial velocity $v_i$
of a projectile (often referred to as the **muzzle velocity**) is increased while keeping $\alpha$ constant, the range increases. This is not surprising, but it is not quite so obvious that the range depends on the *square* of the muzzle velocity and therefore increases, for example, by a factor of four when the initial velocity is doubled.

If $v_i$ is held constant while $\alpha$ is increased from low values, $R$ increases, as we would expect. But when $\alpha = \pi/4$ or $45^\circ$, $2\alpha$ becomes $\pi/2$ and the sine function attains its largest value of $+1$. As $\alpha$ increases beyond this point, $\sin 2\alpha$ decreases. This means that the horizontal range $R$ must have a maximum value (for a fixed muzzle velocity) at an initial velocity elevation of $45^\circ$.

### 3.1 An Example Problem

The profound implication of the Equations 3.9-3.12 is that the coordinates are independent of one another: what happens in the $x$-direction has no effect on what happens in the $y$-direction, and vice-versa. Thus, a ball tossed straight up air inside of a jet will fall straight back down to to the tosser, not sail to the back of the cabin. So too, a projectile aimed at an object suspended at the same horizontal level will strike the object if the latter is dropped at the same instant the projectile is launched.

What can we expect to happen if the second example is altered slightly: say the projectile aims directly at an object not at the same horizontal level? Again, the launch of the projectile and the dropping of the object are simultaneous.

Suppose that the projectile is launched from the origin $(0, 0)$ with initial velocity $v_i = v_0$ at an angle $\alpha$ with the horizontal, and the object begins to fall from rest from position $(L, h)$ [see Figure 3.2], all at instant $t_i = 0$. In this case, motion in the $x$-direction is uniform while motion in the $y$ direction accelerates at $-g$. For the the projectile, then, $x_{p,i} = 0$, $y_{p,i} = 0$, $\dot{x}_{p,i} = v_0 \cos \alpha$, $\dot{y}_{p,i} = v_0 \sin \alpha$, $\ddot{x}_{p} = 0$, and $\ddot{y}_{p} = -g$. For the object, $x_{o,i} = L$, $y_{o,i} = h$, $\dot{x}_{o,i} = 0$, $\dot{y}_{o,i} = 0$, $\ddot{x}_{o} = 0$, and $\ddot{y}_{o} = -g$.

![Figure 3.2: Projectile at 0 aimed at object A, which falls simultaneously with launch.](image)

The equations for the positions the projectile and the object are adapted from Equa-
3.11,  

\[ x = x_i + \dot{x}_i \Delta t + \frac{1}{2} \ddot{x}(\Delta t)^2 \quad y = y_i + \dot{y}_i \Delta t + \frac{1}{2} \ddot{y}(\Delta t)^2 : \]

\[ x_p = v_0 \cos \alpha t \quad y_p = v_0 \sin \alpha t - \frac{1}{2} gt^2 \]

\[ x_o = L \quad y_o = h - \frac{1}{2} gt^2 = L \tan \alpha - \frac{1}{2} gt^2. \]

The projectile therefore reaches the \( x \)-position of the object after a time interval \( t = L/v_0 \cos \alpha \). The respective \( y \)-positions then are:

\[ y_p = L \tan \alpha - \frac{g}{2v_0^2 \cos^2 \alpha} L^2 \]

\[ y_o = L \tan \alpha - \frac{g}{2v_0^2 \cos^2 \alpha} L^2. \]

Thus, \( y_p = y_o \); the projectile is at the same place at the same instant as the object and therefore strikes it. Depending on the magnitude of \( v_0 \), the impact might occur while the projectile is still rising (\( v_0 \) large), or below the horizontal at some point such as B in Figure 3.2 (\( v_0 \) small). If the projectile is launched from ground level and the ground proceeds outward horizontally from the origin, then \( v_0 \) would have to equal or exceed some critical value \( v_{0, \text{crit}} \) for the projectile to reach the object before it hits the ground short of the object’s \( x \)-location.

\[ y_o = 0 \text{ at clock reading } t = \sqrt{2L \tan \alpha / g}. \quad \text{Again, } x_p = \text{ at clock reading } t = L/v_0 \cos \alpha. \]

The minimum necessary initial velocity \( v_{0, \text{crit}} \) for the projectile to strike the object before itself hitting the ground is determined by setting these times equal to one another:

\[ v_{0, \text{crit}} = \sqrt{\frac{gL}{\sin 2\alpha}}, \]

because \( 2 \sin \alpha \cos \alpha = \sin 2\alpha \).

There is another descriptive interpretation of these results: Motion along a parabolic trajectory may be thought of as a superposition of a uniform motion at velocity \( v_0 \) along straight line \( 0A \) in Figure 3.2 combined with free fall through various distances from this line. We show this interpretation with dotted segments in the figure.
Chapter 4

Introduction to Newton’s Laws of Motion

4.1 Mass

Objects differ in their resistance to changes in their state of rest or of uniform rectilinear motion. This property of resistance is given the name inertia. One way of quantifying inertia—of giving it a number and a unit—is to assign a mass number $m$. Inertia has the dimension of mass $M$.

4.2 Newton’s Laws of Motion

First Law: An object continues in its state of rest or of uniform rectilinear motion unless acted upon by a net external force.

The significance of this law has two parts:

- It defines qualitatively the concept of force, as anything that tends to change the state of an object’s motion.

- It implicitly defines an inertial coordinate system as that reference frame in which objects unaffected by a net external force will be found to be at rest or in uniform rectilinear motion.

Second Law: An object of mass number $m$ subject to various forces, $F_1$, $F_2$, $F_3$,..., accelerates (changes its state of rest or uniform rectilinear motion) according to the relationship

$$\sum_{i=1}^{N} F_i = ma_i$$  \hspace{1cm} (4.1)
where \( \sum_i^N \vec{F}_i = \vec{F}_{\text{net}} \) is the vector sum of all the forces acting on the object, and \( \vec{a} \) is the acceleration. If the resultant net force is \( \vec{0} \), then Newton’s Second Law states that \( \vec{a} = \vec{0} \), implying that the object remains at rest or in uniform rectilinear motion: its rectilinear velocity remains constant in both magnitude and direction.

**Third Law:** If one object exerts a force \( \vec{F}_{1\rightarrow 2} \) on a second object, then, regardless of what other forces may be acting on either or both of the two objects, the second object exerts a force \( \vec{F}_{2\rightarrow 1} \) on the first object equal in magnitude and opposite in direction to the force the first exerts on the second:

\[
\vec{F}_{2\rightarrow 1} = -\vec{F}_{1\rightarrow 2}.
\] (4.2)

Notice that Newton’s Third Law implies that there is no possibility of a single, isolated force; forces always arise in pairs. Nevertheless, it is essential to realize that this does not mean that all forces cancel: each force of a pair acts on a different object. Newton’s Second Law sums forces exerted on an object only, not forces exerted by the object.

### 4.3 Weight

Inertia, quantified in terms of, for example, a mass number, is a property of an object. The object retains this property quantitatively (the mass number is the same) anywhere in space and time as long as the physical properties of the object remain unchanged.

**Weight,** on the other hand, is the name for the gravitational force exerted on the object, and so depends on the nature of the gravitational force at the specific location in space and time the object happens to occupy. Where the gravitational acceleration is \( \vec{g} \), the weight \( \vec{W} \) of an object of mass number \( m \) is:

\[
\vec{W} = m\vec{g}.
\] (4.3)

### 4.4 Inertial and Non-inertial Reference Frames

Consider the following: Sitting in a train, you hold a ball up so as to be visible by an observer standing along the tracks. If the train is moving uniformly in a straight line, you will see the ball at rest in your hand and the observer will see it moving at constant velocity. From both points of view, according to Newton’s Second Law, \( \vec{F}_{\text{net}} = m\vec{a} = \vec{0} \). If, with the train moving uniformly in a straight line, you drop the ball, you will see it accelerate uniformly straight down to the floor of the train. An outside observer will see the ball follow a parabolic trajectory, just like projectile motion: uniform rectilinear motion horizontally and uniformly accelerated rectilinear motion vertically. Again, both of you will agree as to the form and content of Newton’s Second Law: \( F_{x\text{net}} = ma_x = 0 \) and \( F_{y\text{net}} = ma_y = mg \), assuming “down” to be the positive \( y \)-direction. But, if the train were changing its velocity—speeding up, or slowing down, or changing direction, or some combination of these—then there would
4.5. CALCULATING NET FORCES AND ACCELERATIONS

no longer be agreement between you and the observer with regard to Newton’s Second Law. If you were holding the ball, then, regardless of the train’s motion, you would see the ball at rest, so $\vec{F}_{\text{net}} = m\vec{a} = \vec{0}$ for you, but the outside observer would see the ball accelerating along with you and the train, assigning a force, say, from your hand, to be acting on the ball, changing its motion. If, rather, you were to let the ball go as the train’s motion were changing, you’d identify a horizontal component in the ball’s acceleration–backward if the train happened to be speeding up, forward if the training were slowing down, or to the right or left if the train were turning–along with the gravitational acceleration. The outside observer, however, would see the same projectile path as if the train were moving uniformly. You’d have to assign a horizontal component to the force, but the outside observer would not.

Frames of reference in which all observers measure the same behavior (get the same form and content for Newton’s Second Law, for example) move at constant velocity relative to one another and are known as inertial. When observers in two different frames cannot agree, it means that one or both of the frames are accelerating, and the frames are called non-inertial. Determined from a frame accelerating at $\vec{a}_n$, Newton’s Second Law requires an extra term to describe the motion of an object:

$$\sum \vec{F} - m\vec{a}_n = m\frac{d\vec{v}'}{dt},$$

(4.4)

where $d\vec{v}'/dt$ is the acceleration of the object as measured by an observer in the accelerating frame (not equal to $\vec{a}$ measured by an observer in an inertial frame) and $\sum \vec{F}$ is the vector sum of all readily identifiable, physical forces acting on the object. The term $-m\vec{a}_n$ is sometimes interpreted as a force, referred to as an inertial force or fictitious force.

4.5 Calculating Net Forces and Accelerations

A sound, and therefore strongly recommended, approach to calculating net forces and resultant accelerations follows:

1. Draw a reasonably careful picture of the situation.
2. Consider separately (i.e, isolate) the object whose motion is under investigation.
3. Draw as vectors all forces acting on the object, roughly indicating the magnitude and direction of each force (that is, create a so-called free body diagram). Note that the forces will be acting at a point for the idealization of objects as points.
4. Determine the resultant net force, $\vec{F}_{\text{net}}$.
5. Choose a convenient inertial coordinate system, indicating the positive direction(s) explicitly. The earth may be considered inertial for the most part.
6. Apply Newton’s Second Law: $\vec{F}_{\text{net}} = m\vec{a}$, where $\vec{a}$ is relative to the coordinate system you have chosen.
4.6 Some Simple Examples

Example 1 A block with mass number \( m \) is lifted upward with force \( \vec{T} \) [see Figure 4.1] resulting in a vertical acceleration.

\[
\begin{align*}
\vec{T} & \quad \text{lifting force} \\
\vec{W} & = mg \quad \text{weight} \\
\vec{F}_{\text{net}} & \quad \text{net force}
\end{align*}
\]

Figure 4.1: a) Lifted, a block accelerates vertically. b) Free body diagram. c) Vector diagram.

Two forces are acting on the block: \( \vec{T} \), which we show in the free body diagram as an arrow placed along the line of the cord attached to the block (whatever is exerting this force is irrelevant and not shown), and \( \vec{W} \), the weight, which is really the resultant of a great number of parallel, downward forces that the earth exerts on the small chunks that we might visualize make up the block, and shown by means of an arrow passing through some central point where we might think of the block’s inertia as being concentrated. If the block has sizable dimensions and the force \( \vec{T} \) were applied at one corner instead of along the center line, the block would tend to rotate as well as move vertically. In placing \( \vec{W} \) and \( \vec{T} \) along a common line of action, we avoid the complication of rotational effects we are not yet prepared to analyze. In this sense, we are treating a finite object as a particle with only translational “degrees of freedom” of motion.

What could be the direction and magnitude of the vertical acceleration for different values of \( m \) and \( T \)? We introduce the \( x \)- and \( y \)-axes, as shown in the figure, and apply Newton’s Second Law:

\[
F_{\text{x net}} = ma_x, \quad F_{\text{y net}} = ma_y.
\]

From the free body diagram, we see that \( F_{\text{x net}} \) and \( a_x \) are zero, so the horizontal direction is of no further concern. Taking the \( y \)-direction as positive upward, we have

\[
\vec{F}_{\text{y net}} = | \vec{T} | - | \vec{W} | = | \vec{T} | - m | \vec{g} |.
\]

We illustrate graphically this algebraic result in the figure. Substituting the expression for \( F_{\text{y net}} \) into the Second Law equation,

\[
| \vec{T} | - m | \vec{g} | = ma_y,
\]
and solving for the acceleration,

\[ a_y = \frac{|\vec{T}| - m |\vec{g}|}{m}. \]

Therefore, we find:

1. If \(|\vec{T}| > m |\vec{g}|\), then \(a_y\) is positive, indicating an upward acceleration. Naturally, it does not matter whether \(|\vec{T}|\) is applied as a pull from above or as a push from below. To accelerate an object upward, the externally applied upward force must be \textit{larger} than the weight of the object. Increasing \(m\) will decrease the acceleration if \(\vec{T}\) remains fixed.

2. If \(|\vec{T}| = m |\vec{g}|\), the acceleration is zero. This is the situation in which our pull is equal and opposite to the pull of gravity and the effects cancel. The block may be motionless (hanging in the air, for example) or moving up or down at \textit{uniform} velocity. In either case, the net force and acceleration are zero, and the system is said to be \textit{in (static) equilibrium}.

3. If \(|\vec{T}| < m |\vec{g}|\), \(a_y\) is negative, indicating a downward acceleration. If the upward pull is less than the weight, the body accelerates downward.

4. If \(\vec{T} = 0\), \(a_y = -|\vec{g}|\), and the body is in a state of free fall.

If the direction of \(\vec{T}\) were reversed, and the block, instead of being lifted, were pushed downward, the downward acceleration would then have a magnitude larger than \(|\vec{g}|\).

\textbf{Example 2}  
A box, resting on the floor, is pulled upward with a force insufficient to impart vertical acceleration [see Figure 4.2]. It’s not falling, either. But, since we know gravity doesn’t turn off, we infer that either the upward pull is exactly equal to the gravitation pull or that there is another force perpendicular or \textit{normal} to the floor. To account for all possibilities, we introduce a so-called \textit{normal force} \(N\) which you see included in the figure.

The normal force is a “passive” force which adjusts itself to some value dictated by the \textit{law of motion} (Newton’s Second Law). Such forces are always present whenever objects are pressed together by external actions. You might think of these as arising from the tendency of deformed surfaces to “spring back” to their original shapes.

We ask how \(N\) adjusts itself in our case. Taking the positive direction upward,

\[ F_{y \text{ net}} = |\vec{T}| + |N| - |\vec{W}|. \]

Recall that in this case there is no acceleration, so \(F_{y \text{ net}} = ma_y = 0\). Therefore,

\[ |\vec{T}| + |N| - |\vec{W}| = 0, \]

and, so,

\[ |N| = |\vec{W}| - |\vec{T}| = m |\vec{g}| - |\vec{T}|. \]
1. If $|\vec{T}| = 0$, then the normal force $|\vec{N}| = |\vec{W}| = m|\vec{g}|$ the weight of the box.

2. If $|\vec{T}|$ is increased, then $|\vec{N}|$ becomes smaller and equals zero when $|\vec{T}| = m|\vec{g}|$.

3. If $|\vec{T}| > m|\vec{g}|$ and the acceleration remains zero, then $\vec{N}$ must be directed downward, that is, the box must be secured (say, glued or nailed) to the floor.

**Example 3** A frictionless disk with mass number $m$ on a plane inclined at an angle $\theta$ from the horizontal is acted upon by a force $\vec{P}$ parallel to the plane [see Figure 4.3].

The forces in this example are not all collinear. We take them to pass through a single point, treating the extend object as a point particle, and we thereby ignore tipping or rotating effects. As in Example 2, the plane exerts a passive normal force. For convenience, we set up a reference system so that one axis is parallel to the direction of the acceleration, and the other axis perpendicular to it. In so doing, $\vec{P}$ is also parallel to an axis and the normal force $\vec{N}$ is perpendicular. The weight $\vec{W}$ must be decomposed into components. Plane geometry tells us that the component of $\vec{W}$
parallel to the plane $W_x = -mg \sin \theta$, and the component perpendicular to the plane $W_y = -mg \cos \theta$. Then,

\[
F_{x \, \text{net}} = |\vec{P}| - mg \sin \theta = ma_x
\]
\[
F_{y \, \text{net}} = |\vec{N}| - mg \cos \theta = 0.
\]

Therefore,

\[
a_x = \frac{|\vec{P}| - mg \sin \theta}{m}
\]
\[
|\vec{N}| = mg \cos \theta.
\]
Chapter 5

Curved Trajectories in a Plane

5.1 Plane Polar Coordinates

Up to this point, we’ve been employing what are called rectangular or Cartesian coordinate systems to define our spaces. We now introduce another sort of coordinate system, using plane polar coordinates, which for some purposes, like curved motion in a plane, is much more convenient.

Plane polar coordinates include measures of the spatial displacement from an origin $\vec{\rho}$ and of the angular displacement $\vec{\theta}$ from a reference line, typically the horizontal [see Figure 5.1]. The dimension of $\vec{\rho}$ is, of course, length $L$. The angle is a dimensionless variable (as an argument of a transcendental function must be), nevertheless, it is measured in a unit given by the dimensionless ratio of the arc length subtended to the radius of the circle inscribed called the radian; there are $2\pi$ radians in a circle, since the circumference of a circle is given by $2\pi r$, where $r$ is the radius of the circle.

![Figure 5.1: A circular trajectory in a) Cartesian and b) plane polar coordinates.](image)
5.2. ANGULAR MOTION

Naturally, the physics cannot depend on the coordinate system, and so it must be that the coordinate systems can easily be related (transformed into) one another. The relationship is specified in Table 5.1.

<table>
<thead>
<tr>
<th>Cartesian Coordinates</th>
<th>Plane Polar Coordinates</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x =</td>
<td>\vec{\rho}</td>
</tr>
<tr>
<td>$y =</td>
<td>\vec{\rho}</td>
</tr>
</tbody>
</table>

Table 5.1: Transformation rules between Cartesian and Plane Polar Coordinates.

5.2 Angular Motion

In plane polar coordinates, motion is described with angular variables:

Angular displacement:

$$\Delta \vec{\theta} \equiv \vec{\theta}_2 - \vec{\theta}_1$$  \hspace{1cm} (5.1)

Angular velocity:

$$\vec{\omega} \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{\theta}}{\Delta t} = \frac{d\vec{\theta}}{dt}$$  \hspace{1cm} (5.2)

Angular acceleration:

$$\vec{\alpha} \equiv \lim_{\Delta t \to 0} \frac{\Delta \vec{\omega}}{\Delta t} = \frac{d\vec{\omega}}{dt} = \frac{d^2\vec{\theta}}{dt^2}$$  \hspace{1cm} (5.3)

Notice that, in terms of dimensions, $\Delta \vec{\theta}$ is dimensionless, while $[\vec{\omega}] = T^{-1}$ and $[\vec{\alpha}] = T^{-2}$.

The direction of each vector is determined by a right-hand rule as in Figure 1.5, where the fingers follow the direction of the motion in the cases of angular displacement and velocity or of change in angular velocity in the case of angular acceleration (if the angular velocity is increasing, then the angular acceleration is in the same direction as the motion; if the angular velocity is decreasing, then the angular acceleration points opposite to the motion). Notice, for each variable, the direction is perpendicular to the plane of the motion.

Note that these variables are applicable to both the motion of an object in translation and that of an object rotating on an axis.

5.3 Uniform Angular Motion

Consider an object in uniform angular motion, that is, with constant angular velocity $\vec{\omega}$ in a circle of constant radius $\vec{\rho}$ [see Figure 5.2]. The frequency $f$ of the object’s motion is defined as the number of times the particle passes a given point in one unit of time. It is related to the angular velocity by:
\[ f = \frac{|\vec{\omega}|}{2\pi}. \] (5.4)

The dimension of frequency is \( T^{-1} \).

The period \( T \), the interval of time it takes for the particle to make one complete cycle, is the reciprocal of the frequency:

\[ T = \frac{1}{f} = \frac{2\pi}{|\vec{\omega}|}. \] (5.5)

The dimension of the period is obviously \( T \).

If, as in Figure 5.2 b), the particle starts with an initial angular position \( \vec{\theta}_i \), then, after time interval \( \Delta t \), the angular position will be

\[ \vec{\theta} = \vec{\theta}_i + \vec{\omega}\Delta t. \] (5.6)

Be careful to notice that Equation 5.6 holds only for the case of uniform (constant) angular velocity.

It is important to appreciate that, while the angular velocity may be constant, which means that the angular acceleration must be zero, the particle’s rectilinear acceleration cannot be zero; the direction of the motion is changing continuously. We determine this acceleration by first relating the angular and rectilinear velocities to one another.

Recall from Table 5.1,

\[ x = |\vec{\rho}| \cos |\vec{\theta}| \quad y = |\vec{\rho}| \sin |\vec{\theta}|. \]

From Figure 5.2, we see that

\[ \vec{\theta} = \Delta \vec{\theta} + \vec{\theta}_i = \vec{\omega}\Delta t + \vec{\theta}_i. \]
5.3. **UNIFORM ANGULAR MOTION**

In this case, since the vectors are parallel (as you’ll see shortly), \( |\vec{\theta}| = |\Delta \vec{\theta} + \vec{\theta}_i| = |\Delta \vec{\theta}| + |\vec{\theta}_i| = |\vec{\omega}||\Delta t + \vec{\theta}_i| \). So,

\[
\begin{align*}
  x &= |\vec{\rho}| \cos(|\vec{\omega}|(\Delta t + |\vec{\theta}_i|)) = |\vec{\rho}| \cos(|\vec{\omega}|(|t - t_i| + |\vec{\theta}_i|)) \\
  y &= |\vec{\rho}| \sin(|\vec{\omega}|(\Delta t + |\vec{\theta}_i|)) = |\vec{\rho}| \sin(|\vec{\omega}|(|t - t_i| + |\vec{\theta}_i|)).
\end{align*}
\]

(5.7)

Recall that the vector \( \vec{\rho} = (x, y) \), and has magnitude

\[
|\vec{\rho}| = \sqrt{x^2 + y^2},
\]

and direction

\[
\tan|\vec{\theta}| = \tan(|\vec{\omega}|(\Delta t + |\vec{\theta}_i|)) = \frac{y}{x}.
\]

Taking the derivative with respect to time of Equations 5.7,

\[
\begin{align*}
  \dot{x} &= -|\vec{\omega}||\vec{\rho}| \sin(|\vec{\omega}|(|t - t_i| + |\vec{\theta}_i|)) = -|\vec{\omega}|y \\
  \dot{y} &= |\vec{\omega}||\vec{\rho}| \cos(|\vec{\omega}|(|t - t_i| + |\vec{\theta}_i|)) = |\vec{\omega}|x.
\end{align*}
\]

(5.8)

Thus, the rectilinear velocity, \( \vec{v} = (\dot{x}, \dot{y}) \), has magnitude

\[
|\vec{v}| = \sqrt{\dot{x}^2 + \dot{y}^2} = |\vec{\omega}||\sqrt{\dot{x}^2 + \dot{y}^2} = |\vec{\omega}||\vec{\rho}|,
\]

(5.9)

and direction

\[
\frac{\dot{y}}{\dot{x}} = -\frac{x}{y}.
\]

(5.10)

Notice that the direction of the velocity \( \vec{v} \) is the negative reciprocal of the direction of the position vector \( \vec{\rho} \). Lines with slopes that are negative reciprocals of one another are perpendicular. Therefore, in the case of uniform circular motion, the rectilinear velocity is always perpendicular to the position vector.

Notice, too, that the magnitude \( |\vec{v}| = |\vec{\rho}||\vec{\omega}| \) is proportional to both the radius of the circle and the angular velocity. The magnitude of an object’s rectilinear velocity remains always unchanged if it moves in a circle at constant angular velocity, but two objects going circles of different sizes, one large and one small, with the same angular velocity have different (though still constant) rectilinear velocities: the rectilinear velocity of the object moving in the larger circle is greater than the rectilinear velocity of the object going in the smaller circle. An ant on the outer portion of a CD gets a faster ride than an ant near the center of the same CD.

Taking the derivatives now of Equations 5.8 with respect to time

\[
\begin{align*}
  \ddot{x} &= -|\vec{\omega}|^2|\vec{\rho}| \cos(|\vec{\omega}|(|t - t_i| + |\vec{\theta}_i|)) = -|\vec{\omega}|^2x \\
  \ddot{y} &= -|\vec{\omega}|^2|\vec{\rho}| \sin(|\vec{\omega}|(|t - t_i| + |\vec{\theta}_i|)) = -|\vec{\omega}|^2y.
\end{align*}
\]

(5.11)
CHAPTER 5. CURVED TRAJECTORIES IN A PLANE

Now, \( \vec{a} = (\ddot{x}, \ddot{y}) = -|\vec{\omega}|^2 (x, y) \), or
\[
\vec{a}_\rho = -|\vec{\omega}|^2 \hat{\rho}.
\]
(5.12)

The acceleration is anti-parallel to the position vector. That is, the acceleration during uniform circular motion continuously points inward toward the center of the circle. This radial acceleration is frequently referred to as centripetal acceleration and identified with a subscript, here \( \rho \).

Also, since \( \hat{\rho} = \frac{\vec{\rho}}{|\vec{\rho}|} \) and, by equation 5.9, \( |\vec{v}| = |\vec{\omega}| |\vec{\rho}| \), equation 5.12 may be written
\[
\vec{a}_\rho = -|\vec{\omega}|^2 \hat{\rho} = -|\vec{v}|^2 \hat{\rho}.
\]

5.4 Angular Motion with Varying Angular Velocity

Recall,

The equation, 5.1, for angular displacement: \( \Delta \theta = \theta_2 - \theta_1 \)

The equation, 5.2, for angular velocity: \( \vec{\omega} = \frac{d\vec{\theta}}{dt} \)

The equation, 5.3, for angular acceleration: \( \vec{\alpha} = \frac{d^2\vec{\theta}}{dt^2} = \frac{d^2\theta}{dt^2} \).

5.4.1 Uniform Angular Acceleration

The formulas which relate angular displacement, angular velocity, and angular acceleration to one another and to changes in time are exactly analogous to those for uniformly accelerated rectilinear motion [see Table 5.2]. Initial values are indicated with the subscript \( i \).

<table>
<thead>
<tr>
<th>Uniformly Accelerated Rectilinear Motion</th>
<th>Uniformly Accelerated Angular Motion</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \vec{v} = \vec{v}_i + \vec{a}\Delta t )</td>
<td>( \vec{\omega} = \vec{\omega}_i + \vec{\alpha}\Delta t )</td>
</tr>
<tr>
<td>( \vec{\omega} = \frac{1}{2}(\vec{v} + \vec{v}_i) )</td>
<td>( \vec{\omega} = \frac{1}{2}(\vec{v} + \vec{v}_i) )</td>
</tr>
<tr>
<td>( \vec{r} = \vec{r}_i + \frac{1}{2}\vec{v}\Delta t + \frac{1}{2}\vec{a}(\Delta t)^2 )</td>
<td>( \vec{\theta} = \vec{\theta}_i + \vec{\omega}_i\Delta t + \frac{1}{2}\vec{\alpha}(\Delta t)^2 )</td>
</tr>
<tr>
<td>(</td>
<td>\vec{v}</td>
</tr>
</tbody>
</table>

Table 5.2: Kinematic equations for uniformly accelerated rectilinear and angular motion.

Notice, in the last set of equations, the dot product indicates that only the component of the acceleration (and therefore the force) parallel to the motion changes the magnitude of the velocity. The component(s) of acceleration perpendicular to the motion change(s) only the motion’s direction, as we’ll see now.
5.4. ANGULAR MOTION WITH VARYING ANGULAR VELOCITY

5.4.2 Circular Motion

When moving in a circle, an object remains at a constant distance \( \| \vec{\rho} \| \) from a fixed axis of rotation [see Figure 5.2]. You know that the circumference of a circle is related to the radius of the circle by

\[
C = 2\pi \| \vec{\rho} \|,
\]

where, as we discussed above, \( 2\pi \) is the total angle measure (in radians) of the circle. Any arc length \( \Delta \vec{r} \), or portion of the circumference, must be similarly related to the radius and angle \( \Delta \vec{\theta} \) that subtends it:

\[
| \Delta \vec{r} | = | \Delta \vec{\theta} | \| \vec{\rho} \|. \tag{5.13}
\]

This relationship allows us to relate rectilinear quantities to angular quantities in circular motion.

If we divide both sides of Equation 5.13 by the time interval \( \Delta t \) and take the limit as \( \Delta t \to 0 \), we get the time derivatives:

\[
\lim_{\Delta t \to 0} \frac{| \Delta \vec{r} |}{\Delta t} = \frac{\Delta \vec{\theta}}{\Delta t} \| \vec{\rho} \|
\]

\[
\left| \frac{d\vec{r}}{dt} \right| = \left| \frac{d\vec{\theta}}{dt} \| \vec{\rho} \| \right|
\]

\[
\left| \vec{v} \right| = \left| \vec{\omega} \| \vec{\rho} \| \right|,
\tag{5.14}
\]

where, in this instance, \( \vec{\rho} \) is constant and so unaffected by the limit.

This is a relationship between the magnitudes of rectilinear and angular velocities in circular motion. Referring to Figure 5.2 and recalling our convention for the direction of angular variables, we note that each vector in Equation 5.14 is perpendicular to the other two vectors: in our example, \( \vec{\omega} \) points out of the page, \( \vec{\rho} \) points to the top-right of the page, and \( \vec{v} \) points to the top left. This kind of relationship should remind you of the vector or cross-product, Equation 1.15 and Figure 1.5. With reference to the right-hand rule, we can rewrite Equations 5.13 and 5.14 to include both magnitude and direction:

\[
\Delta \vec{r} = \Delta \vec{\theta} \times \vec{\rho} \tag{5.15}
\]

\[
\vec{v} = \vec{\omega} \times \vec{\rho}. \tag{5.16}
\]

Similarly, taking the time derivative of Equation 5.14 or 5.16,

\[
\vec{a}_t = \vec{\alpha} \times \vec{\rho}. \tag{5.17}
\]

Notice that we’ve given the subscript \( t \) to the rectilinear acceleration. It’s direction is perpendicular to both \( \vec{\alpha} \), which points into the page (if the rotation is slowing down) or out of the page (if it is speeding up), and \( \vec{\rho} \), which is radial. That is, this acceleration must point tangent to the motion, either parallel or anti-parallel to the velocity.
46

CHAPTER 5. CURVED TRAJECTORIES IN A PLANE

Thus, in circular motion, there are two accelerations: a radial or centripetal acceleration and a tangential acceleration. The total acceleration is the vector sum of these two [see Figure 5.3]:

\[ \vec{a} = \vec{a}_\rho + \vec{a}_t, \]  

(5.18)

Hence, as an object in circular motion changes angular velocity, without changing its radius of rotation, the acceleration points neither just radially nor just tangentially, but somewhere between.

![Figure 5.3: General motion in a plane with instantaneous curvature of radius $\vec{\rho}$ around an axis through c.](image)

5.5 Motion Along a General Plane Curve

Putting all of this together: an object moving through space with position as a function of time $\vec{r} = \vec{r}(t)$ relative to some origin [see Figure 5.3, where the origin is not shown], has

an instantaneous velocity: $\vec{v} = \frac{d\vec{r}}{dt}$,

a tangential component of the instantaneous acceleration: $\vec{a}_t = \frac{d^2\vec{r}}{dt^2}$, and

a radial component of the instantaneous acceleration: $\vec{a}_\rho = -\frac{1}{|\vec{\omega}|^2} |\vec{\omega}| \times |\vec{\rho}| \hat{\rho}$,

where $\vec{\omega}$ is the instantaneous angular velocity around an axis through the instantaneous center of curvature of radius $\vec{\rho}$, $|\vec{v}| = |\vec{\omega} \times \vec{\rho}| = |\vec{\omega}| |\vec{\rho}|$ (because instantaneously, by construction, $\vec{\omega}$ and $\vec{\rho}$ are perpendicular), and $\hat{\rho}$ is the unit vector in the direction of $\vec{\rho}$.

These results, along with the vector form of Newton’s Second Law, $\vec{F}_{\text{net}} = m\vec{a}$, imply that the resultant net force has a component parallel or anti-parallel to the radius of curvature $\vec{\rho}$ and one or two components perpendicular to $\vec{\rho}$:

\[ \vec{F}_\rho = m\vec{a}_\rho \quad \vec{F}_t = m\vec{a}_t. \]  

(5.19)
If $\vec{F}_\rho > \vec{0}$, then the object accelerates toward the center of curvature and bends around the curve.
Chapter 6

Applications of Newton’s Laws

6.1 Relative Motion

Consider two investigators observing the motion of an object. Person 1 measures the object to be moving at velocity \( \vec{v}_1 \) and person 2 to be moving at velocity \( \vec{u} \). Person 2 measures the object to be moving at velocity \( \vec{v}_2 \). **Galilean** or **Newtonian relativity** relates these velocities:

\[
\vec{v}_1 = \vec{u} + \vec{v}_2.
\]

(6.1)

Note that person 2 measures person 1 to be moving at velocity \(-\vec{u}\), so the reverse relationship follows:

\[
\vec{v}_2 = -\vec{u} + \vec{v}_1.
\]

(6.2)

If person 1 is at rest or in uniform motion relative to an inertial frame of reference—that is, if the reference frame of person 1 is inertial, then, if \( \vec{u} \) is constant, person 2’s frame of reference is also inertial, and both investigators will measure the same value for the object’s acceleration.

6.2 Center of Mass and Systems of Interacting Particles

The **center of mass** \( CM \) of an extended object is the position at which the object responds to a net external force just as a point particle with the same **mass number** would if it were located at this position:

\[
\vec{F}_{net} = \sum \vec{F}_{ext} = M \vec{a}_{CM},
\]

(6.3)

where \( M \) is the mass number of the extended object and \( \vec{a}_{CM} \) is the acceleration of the center of mass.

An extended object can be considered a system of particles, however these may be distinguished. It is important, when applying Equation 6.3 to a complex system,
6.3 NORMAL AND FRICTIONAL FORCES

to clearly define which particles are in the system and which are outside, because, sometimes, what is called “an object” or a system, has no clear boundary. The mass number of such a system is the sum of the mass numbers of its constituent particles (mass number, a positive-definite scalar, is simply additive):

$$M = m_1 + m_2 + \cdots + m_N = \sum_{i=1}^{N} m_i,$$

(6.4)

where the subscript $i$ indicates the $i$th particle.

Each particle is at position $\vec{r}_i$. Then the center of mass is defined:

$$M\vec{r}_{CM} \equiv m_1\vec{r}_1 + m_2\vec{r}_2 + \cdots + m_N\vec{r}_N$$

or

$$\vec{r}_{CM} \equiv \frac{1}{M} \sum_{i=1}^{N} m_i\vec{r}_i.$$

(6.5)

The center of gravity of an extended object is the position at which the object responds to the force of gravity just as a point particle with the same weight would if it were located at this position. Thus, we can consider the weight of an extended object as a force acting on the object’s center of gravity and not worry about its effect on individual constituents. An extended, rigid object can be suspended in any orientation by its center of gravity without tending to rotate. If the gravitational force (the field) is uniform (the same everywhere in the neighborhood), the center of gravity and the center of mass coincide.

### 6.3 Normal and Frictional Forces

An object in contact with another object exerts a force on the second object (and vice-versa) that has a component normal (perpendicular) to the surfaces of contact, the normal force $\vec{N}$, and a component parallel to the surfaces of contact. The parallel component is called the force of friction and is usually symbolized $\vec{f}$, with a subscript $k$ or $s$ (to be defined shortly) indicating the kind of frictional force involved. Frictional forces are directed so as to oppose the motion, that is, to oppose the sliding or the tendency to slide of one surface relative to another.

Experiment has shown that the magnitudes of the frictional and normal forces are roughly proportional to one on another. This proportionality is clearest when the surfaces of objects slide against one another. The frictional force in this case is called kinetic friction, and the relationship between the force of kinetic friction and the normal force is given approximately by:

$$|\vec{f}_k| = \mu_k |\vec{N}|,$$

(6.6)

where the subscript $k$ indicates kinetic friction, or the friction associated with sliding. $\mu_k$ is the coefficient of kinetic friction; its size determines the magnitude of the frictional force for a given normal force.
When surfaces are at rest relative to one another, the parallel component of the force is known as **static friction**. Static friction is a “passive” force, like the normal force: in the presence of an external force also parallel to the surfaces, the force of static friction varies in opposition, remaining equal in magnitude and preventing motion. However, should the external force increase to a certain limit beyond which the force of static friction cannot exceed, the surfaces will begin to slide relative to one another. Once this occurs, the force of kinetic friction is found to remain essentially constant, but at magnitude less than the maximum value static friction reached before giving out.

The magnitude of the force of static friction then varies, depending on the presence of external forces parallel to the surface, between zero and some maximum found by experiment to be proportional to the normal force. We may write this as:

\[ | \vec{f}_s | \leq \mu_s | \vec{N} | , \]

where the subscript \( s \) indicates **static friction**, and \( \mu_s \) is the **coefficient of static friction**. It characterizes (dependent on the materials and surface features involved) the maximum magnitude of the static friction between two surfaces resisting motion from rest. It is typically larger than \( \mu_k \) for the same surfaces in contact.

When one object rolls over another, both surfaces deform at the line or point of contact. The rolling therefore is always slightly uphill, and this resisting force is called **rolling friction**.

### 6.4 Elasticity and Hooke’s Law

An **elastic** object returns to its original shape and dimensions once a deforming force ceases to exert itself on the object. A deforming force exerts what is called **stress** on an object; stress is defined as the ratio of the magnitude of the force to the surface area of the object over which the force is exerted. It is another name for **pressure**. For an approximately one dimensional object (like a string), one can think of stress as a tension force/unit length. The result of stress is **strain**, which may take three forms:

- **Linear strain** is the ratio of the change in length of the object to its total length, \( \Delta L / L \).
- **Volume strain** is the ratio of the change in volume of the object to its total volume, \( \Delta V / V \).
- **Shear** is the angular change \( \tan \theta \approx \theta \) in shape of the object from its normal shape.

**Hooke’s Law** states that, for elastic objects, as long as a deformation remains within the object’s so-called **elastic limits** (limits of deformation beyond which the object loses its elastic properties—it stays deformed), the strain is proportional to the stress:

\[ \text{stress} = (\text{elastic modulus})(\text{strain}) , \]
where an elastic modulus characterizes the degree of proportionality for different objects. For small deformations, elastic moduli are essentially constant.

The elasticity of an object can be attributed to a restoring force, which typically is a manifestation of electrostatic repulsion (but this is beyond our discussion here). In any case, the standard model of this is the spring. The ensuing behavior of objects slightly displaced from equilibrium is well approximated by employing this model, in which the strength of the restoring force of the spring is proportional to its contraction or elongation. In one dimension, if a particle with mass number \( m \) is secured to the end of a spring of length \( \ell \) and displaced from the equilibrium position by a distance \( \Delta \ell \), then

\[
\frac{F_{\text{res}}}{\ell} = -k \frac{\Delta \ell}{\ell},
\]

\[
F_{\text{res}} = -k \Delta \ell, \tag{6.8}
\]

where \( k \) is the elastic modulus of the spring, also known as the spring constant, which gives the magnitude of the restoring force for one unit of displacement from equilibrium. The minus sign indicates that the force always points in the direction opposite to the displacement and toward to the equilibrium position.

### 6.5 Uniform Circular Motion

Recall that an object with mass number \( m \) moving in a circle of radius \( |\vec{\rho}| \) at constant angular velocity \( \vec{\omega} \) accelerates centripetally (radially toward the axis of rotation):

Equation 5.12: \( \vec{a}_\rho = -|\vec{\omega}|^2 \vec{\rho} = -\frac{|\vec{v}|^2}{|\vec{\rho}|} \hat{\rho}. \)

\( \vec{v} = \vec{\omega} \times \vec{\rho} \) is the object’s instantaneous rectilinear velocity.

By Newton’s Second Law, there must be a net external force acting radially inward toward the center of the circle:

\[
\vec{F}_{\text{net}} = \sum \vec{F}_{\text{ext}} = -m |\vec{\omega}|^2 \vec{\rho} = -\frac{m |\vec{v}|^2}{|\vec{\rho}|} \hat{\rho}. \tag{6.9}
\]

Whatever the source or sources—whether the tension in a string, the elastic force of a spring, the normal force of the wall of a tube, the force of static friction between tire and road–forces that collectively or individually induce centripetal acceleration are labeled centripetal forces. Forces may act centripetally; but there is no such thing in and of itself as a centripetal force.

### 6.6 Gravitational Force

We saw earlier that the force of gravity manifests as free fall acceleration \( g \) and weight \( m g \). Newton proposed a Universal Law of Gravitation stating that any two particles,
CHAPTER 6. APPLICATIONS OF NEWTON’S LAWS

with mass numbers, say, \( m_1 \) and \( m_2 \), separated by distance \( \vec{r} \), exert an attractive force on one another:

\[
\vec{F}_G = -G \frac{m_1 m_2}{|\vec{r}|^2} \hat{r},
\]

where \( G \) is called the gravitational constant, which is universal, meaning it has the same value (depending on units) everywhere in the universe. It’s a very small number, indicating the gravitational force is exceedingly weak. But because gravitation is always attractive, and so agglomeration is a striking feature of the universe, gravitation is the force we are most familiar with.

Let us identify \( m_2 \) with the earth’s mass number \( m_e \), \( m_1 \) with the mass number \( m \) of some object near the earth’s surface, and \( |\vec{r}| \) with the radius of the earth \( r_e \), and then rewrite Equation 6.10 in the familiar form of Newton’s Second Law:

\[
|\vec{F}_G| = G \frac{m m_e}{r_e^2} = m |\vec{a}|,
\]

\[
|\vec{a}| = G \frac{m_e}{r_e^2}.
\]

But we symbolize the acceleration due to gravity as \( \vec{g} \), and so we get:

\[
|\vec{g}| = G \frac{m_e}{r_e^2}.
\]

In this form, \( \vec{g} \) is also known as the field intensity at a distance \( r_e \) from the center of the earth, which is the source of a gravitational field associated with its inertia. A spherically symmetric distribution of matter gives rise to a gravitational field that affects other objects outside this distribution just as if all the matter were concentrated at the position of its center. Thus, for many typical purposes, we can treat the sun, earth, moon, etc., as point objects and not be concerned that they are in fact extended objects.
Chapter 7

Momentum

7.1 Linear Momentum

An object with mass number \( m \) moving with rectilinear velocity \( \vec{v} \) is said to have momentum

\[
\vec{p} \equiv m\vec{v},
\]

which is obviously a vector quantity with direction the same as the velocity. Its dimensions are \( MLT^{-1} \).

Since \( \vec{a} = \frac{d\vec{v}}{dt} \), Newton’s Second Law may be rewritten

\[
\vec{F}_{\text{net}} = \sum \vec{F}_{\text{ext}} = \frac{d\vec{p}}{dt},
\]

which is actually closer to the way Newton himself stated the law. As such, Equation 7.2 is more general than Equation 4.1, \( \vec{F}_{\text{net}} = m\vec{a} \). This latter follows from the former in case the mass of the system is a constant, but it need not be. This more general form of the Second Law states that the effect of a net external force on an object is to change its momentum, whether this be its velocity or its mass or both.

The total momentum \( \vec{p}_{\text{tot}} \) of a system of particles is the vector sum of each particle’s momentum vector:

\[
\vec{p}_{\text{tot}} = \vec{p}_1 + \vec{p}_2 + \cdots + \vec{p}_N = M\vec{v}_{CM},
\]

a result that follows directly from taking the time derivative of the definition of center of mass, Equation 6.3. Newton’s Second Law for this system becomes

\[
\vec{F}_{\text{net}} = \sum \vec{F}_{\text{ext}} = \frac{d\vec{p}_{\text{tot}}}{dt}.
\]

7.2 Impulse

Multiplying both sides of Equation 7.2 by \( dt \) and integrating.
CHAPTER 7. MOMENTUM

\[ \int_{t_i}^{t} \vec{F}_{\text{net}}(t') \, dt' = \int_{\vec{p}_i}^{\vec{p}} d\vec{p}', \]

or

\[ \int_{t_i}^{t} \vec{F}_{\text{net}}(t') \, dt' = \vec{p} - \vec{p}_i = \Delta \vec{p}, \quad (7.5) \]

where the \( t' \) indicates a dummy variable to be integrated over. This result says that the time-integrated net force is numerically equal to the change in the momentum. This time-integrated force is known as the impulse.

We may replace a time-varying force \( \vec{F}(t) \) by it’s time-averaged value \( \vec{F} \) (that is, by some constant force equal to the average force acting during the time interval under consideration):

\[ \int_{t_i}^{t} \vec{F}_{\text{net}}(t') \, dt' = \vec{F}_{\text{net}} \Delta t = \Delta \vec{p}. \quad (7.6) \]

Measuring the momentum change that occurs in an interaction and its duration permits us to say something about the average force of the interaction, even if we can’t describe it in complete detail.

7.3 Conservation of Linear Momentum

Obviously, from Equation 7.4, if \( \vec{F}_{\text{net}} = \sum \vec{F}_{\text{ext}} = \vec{0} \), then \( \frac{d\vec{p}}{dt} = \vec{0} \). That is, in the absence of a net external force, the momentum remains constant in time. This is a statement of the Law of Conservation of Linear Momentum. It implies that for any system closed to external influence \( \vec{p}_{\text{tot}} = \vec{p}_{\text{tot} i} \), even if the momenta of individual constituents of the system change.
Chapter 8

Introduction to Energy Concepts

8.1 Work and Kinetic Energy

A net external force acting on an object changes its motion: this from Newton’s First and Second Laws. A component of the net force in the direction of the object’s displacement changes the magnitude of the object’s velocity but not its direction; components perpendicular to the object’s displacement change its direction but not its magnitude. The general result, then, of a net external force is to change both magnitude and direction of an object’s velocity.

The product of an object’s mass number, a scalar, and its velocity, a vector, is a vector called momentum, \( \vec{p} = m \vec{v} \). A net external force acting on the object for some interval of time changes its momentum, both magnitude and direction:

\[
\text{Equation 7.5: } \int_{t_i}^{t_f} \vec{F}_{\text{net}}(t') dt' = \Delta \vec{p},
\]

and we say that this change in momentum is equal to the integrated force as a function of time, or impulse.

If, rather, we concern ourselves only with the magnitude of the velocity change, we may begin with the squared velocity, as given in the kinematic equation:

\[
\text{Equation 3.9: } |\vec{v}|^2 = |\vec{v}_i|^2 + 2\vec{a} \cdot \Delta \vec{r}.
\]

Since \( \vec{a} = \vec{F}_{\text{net}} / m \) by Newton’s Second Law, we can rearrange things:

\[
\vec{F}_{\text{net}} \cdot \Delta \vec{r} = \frac{1}{2} m (|\vec{v}|^2 - |\vec{v}_i|^2).
\]

Since \( \Delta \vec{r} = \int_{\vec{r}_i}^{\vec{r}_f} d\vec{r}' \),

\[
\int_{\vec{r}_i}^{\vec{r}_f} \vec{F}_{\text{net}} \cdot d\vec{r}' = \frac{1}{2} m (|\vec{v}|^2 - |\vec{v}_i|^2). \tag{8.1}
\]
The left-hand side of Equation 8.1 is known as work. We say that the net force $\vec{F}_{\text{net}}$ does work on an object with mass number $m$ as it moves from position $\vec{r}_i$ to position $\vec{r}$. Work is a scalar quantity (the result of a dot or scalar product) and has dimension $ML^2T^{-2}$. It is given the symbol $W_{\vec{r}_i\vec{r}}$.

Since, obviously, $W = \int dW$ and $d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$, we can write, for the differential work:

$$dW = ||\vec{F}_{\text{net}}|| |d\vec{r}| \cos \theta = F_x dx + F_y dy + F_z dz,$$

(8.2)

where $\theta$ is the angle between the net force vector and the displacement vector.

The right-hand side of Equation 8.1 has the form $\frac{1}{2}m |\vec{v}|^2$, which is called the kinetic energy and given the symbol $K$. Note that it has, as it must, the same dimensions as work $ML^2T^{-2}$. The right-hand side of 8.1 may therefore be rewritten as $\frac{1}{2}m(|\vec{v}|^2 - |\vec{v}_i|^2) = K - K_i = \Delta K$. This leads to what is known as the Work-Kinetic Energy Principle:

$$W_{\vec{r}_i\vec{r}} = \Delta K,$$

(8.3)

The work done by a net force acting on an object is numerically equal to the change in the object’s kinetic energy.

8.1.1 An Example

Consider an object with mass number $m$ attached to the end of a spring with spring constant $k$. The spring and object lie on a smooth, horizontal surface. Let the position of the object when the spring is in its relaxed configuration be labeled $x_0$, and say the object is displaced from this position to position $x_i$. The object is then released from rest. We ask for the object’s kinetic energy at some position $x$. Using the definition of Work and Work-Kinetic Energy Principle, Equation 8.3,

$$W_{x, x_i} = \int_{x_i - x_0}^{x - x_0} -k\Delta \ell \ d(\Delta \ell)$$

$$= -\frac{1}{2} k(\Delta \ell)^2 \bigg|_{x_i - x_0}^{x - x_0}$$

$$= -\frac{1}{2} k(x - x_0)^2 + \frac{1}{2} k(x_i - x_0)^2 = \frac{1}{2} m |\vec{v}|^2,$$

since $\vec{v}_i = 0$. Experience shows that $|x - x_0| \leq |x_i - x_0|$, so that the left-hand side of this equation is always positive or zero, as the right-hand side must be.

8.2 Power

Power is the instantaneous rate at which work is done:

$$P \equiv \frac{dW}{dt} = \vec{F}_{\text{net}} \cdot \frac{d\vec{r}}{dt} = \vec{F}_{\text{net}} \cdot \vec{v} = ||\vec{F}_{\text{net}}|| |\vec{v}| \cos \theta = F_x \frac{dx}{dt} + F_y \frac{dy}{dt} + F_z \frac{dz}{dt},$$

(8.4)
where $\theta$ is the angle between the instantaneous net force $\vec{F}_{\text{net}}$ and the instantaneous velocity $\vec{v}$.

If the force is constant, then $P = W/\Delta t$.

The dimensions of power are $ML^2T^{-3}$. 
Chapter 9

Potential Energy and Energy Conservation

9.1 Potential Energy

Suppose we lift at constant velocity an object with mass number \( m \) straight up from the floor to a height \( h \). Because the velocity isn’t changing, the acceleration is zero. Consequently, the net force, which is the vector sum of our lifting \( \vec{F} \), say, and the weight \( m\vec{g} \), is zero, as well. Yet, we certainly feel like we do something.

Now, if, after raising the object to height \( h \), we release it from rest, it accelerates downward, attaining a velocity of

\[
| \vec{v} |^2 = | \vec{v}_i |^2 + 2\vec{a} \cdot \Delta\vec{r} \\
| \vec{v} |^2 = 2(-| \vec{g} |)(-h) \\
| \vec{v} | = \sqrt{2| \vec{g} | h}
\]

just as it reaches the floor. We used here Equation 2.14.

The net force after the release is \( F_{net} = m\vec{g} \), so the work done by gravity on the way down is

\[
W_{h0}^{\text{G}} = \int_h^0 m\vec{g} \cdot d\vec{r} = m | \vec{g} | h = \frac{1}{2} m | \vec{v} |^2.
\]

You see, we can get \( | \vec{v} | = \sqrt{2| \vec{g} | h} \) this way, too.

The object achieved this velocity, or kinetic energy, after we lifted it up and let it drop; we first “worked against the force of gravity.” You might say our lifting provided the potential for the object, once released from the end-point of the lift, to gain this much kinetic energy.

Suppose we fix one end of a spring with spring constant \( k \) and attach an object with mass number \( m \) to its other end. If we then pull the object horizontally at constant
velocity along a frictionless surface, displacing it $\Delta \ell_i$, we also stretch the spring $\Delta \ell_i$ from its equilibrium configuration (that is, neither stretched nor compressed). Because the velocity isn’t changing, the acceleration is zero. Consequently, the net force, which is the vector sum of our pull $\vec{F}$, say, and the elastic force (Hooke’s Law) $-k\Delta \ell_i$, is zero, as well. Yet, we certainly feel like we do something.

Now, if, after stretching the spring $\Delta \ell_i$, we release the object from rest, it accelerates back toward its equilibrium position. We cannot use Equation 2.14 to determine the velocity in this case, because the acceleration is not constant, as it is in the case of free fall near the earth’s surface. We saw in the example of the last chapter,

$$W_{\Delta \ell_i, \Delta t} = \int_{\Delta \ell_i}^{\Delta \ell_f} -k\Delta \ell \, d(\Delta \ell) = -\frac{1}{2}k(\Delta \ell)^2 + \frac{1}{2}k(\Delta \ell_i)^2 = \frac{1}{2}mv^2,$$

where $\Delta \ell$ is the object’s later displacement from its position when the spring is relaxed. Thus $|\vec{v}| = \sqrt{\frac{k}{m}[k(\Delta \ell_i)^2 - (\Delta \ell)^2]}$ In this way, we can determine the velocity at any displacement from equilibrium $\Delta \ell$.

But the object only attains a velocity if it is displaced from its equilibrium position. By “working against the spring,” stretching (or compressing) it, provides the object, like the one we lift, with the potential to attain some kinetic energy.

Notice in these cases, lifting against gravity and stretching a spring, that a displacement was necessary to provide the potential for kinetic energy. The potential for kinetic energy due to interaction with a force whose magnitude depends only on position is known, naturally enough, as potential energy. The force involved in the interaction is known as a conservative force. Note that, in both our examples, the route from the initial to final positions makes no difference: for conservative forces, the work is independent of the path joining the endpoints. So, the work is zero for a closed loop. Work done by a non-conservative force (friction, for example) does depend on the route.

In more formal mathematical terms, a conservative force is defined as any force that can be derived from a function of position $U(\vec{r})$, the potential energy, by taking the gradient, or spatial derivative, of this function:

$$\vec{F} = -\vec{\nabla}U(\vec{r}),$$

where $\vec{\nabla} \equiv \frac{\partial}{\partial x} \hat{i} + \frac{\partial}{\partial y} \hat{j} + \frac{\partial}{\partial z} \hat{k}$ and $\frac{\partial}{\partial x}$, etc., are partial derivatives. In this course, we will concern ourselves only with potential functions in 1-dimension, so we can define a conserved force:

$$F_s \equiv -\frac{dU}{ds}, \quad (9.1)$$

where $F_s$ is the component of the force in the direction of $ds$, the infinitesimal displacement from the point of observation. That is, a conservative force is derived by taking the negative derivative with respect to position of the potential energy. Therefore,

$$dU = -\vec{F} \cdot d\vec{r}, \quad (9.2)$$

so that $dW = -dU$. Thus,
Thus, positive (negative) work, which increases (decreases) kinetic energy, can also decrease (increase) potential energy. Remember, when the force and the displacement are in the same direction, the work is positive; when they are in opposite directions, the work is negative. When we lift the ball, the gravitational force is doing negative work, thus increasing the object’s potential energy. One might expect the kinetic energy to decrease accordingly, but our lifting is doing positive work, which would increase kinetic energy, and the net change in kinetic energy is zero. The potential energy increases though because our lifting is not a conservative force (it doesn’t depend simply on position). Gravity is the only conservative force in this interaction, and its (negative) work changes the potential energy. When the object is released to fall freely, the gravitational force does positive work increasing kinetic energy and decreasing potential energy. A similar analysis can be made of the spring stretching.

9.2 Conservation of Mechanical Energy

As long as only conservative forces are involved, then, whenever the kinetic energy increases the potential energy decreases, and vice-versa. This is a statement of the Law of Conservation of Mechanical Energy:

$$\Delta K + \Delta U = 0.$$  \hspace{1cm} (9.4)

This is a special case of a more general Law of Conservation of Energy. If we identify the work done by non-conservative forces \( W' \), then

$$\Delta K + \Delta U = W'.$$ \hspace{1cm} (9.5)

In the lifting example above, the kinetic energy was unchanged during the lift, and the potential energy, increasing against the work of gravity, equals the work done by our lift.
Chapter 10

Torque and Equilibrium

10.1 Rigid Bodies

An object is called a rigid body if the separation between any two constituents of the object is constant. In such a case, the object’s center of mass, \( \mathbf{r}_{CM} = (x_{CM}, y_{CM}, z_{CM}) \), found with Equation 6.5, remains fixed with respect to the object. For real three-dimensional objects, usually the easiest approach to finding the center of mass is to determine components individually:

\[
\begin{align*}
    x_{CM} &= \frac{\sum m_i x_i}{\sum m_i}, \\
    y_{CM} &= \frac{\sum m_i y_i}{\sum m_i}, \\
    z_{CM} &= \frac{\sum m_i z_i}{\sum m_i}.
\end{align*}
\]  

(10.1)

Again, real objects are typically continuous (the constituents are not so easy to isolate), so the sums go over to integrals:

\[
\begin{align*}
    x_{CM} &= \int \frac{x \, dm}{\int dm}, \\
    y_{CM} &= \int \frac{y \, dm}{\int dm}, \\
    z_{CM} &= \int \frac{z \, dm}{\int dm}.
\end{align*}
\]  

(10.2)

10.2 Torque

You will recall the analogy between rectilinear and rotational kinematic equations [see Table 5.2]. The analogy carries over into dynamics, as well. Corresponding to force in rectilinear dynamics is torque in rotational dynamics. Just as a net rectilinear force results in a rectilinear acceleration, a net torque results in angular acceleration.

Torque measures the effectiveness of a force to change an object’s rotation. Experience shows that the magnitude of this rotational acceleration depends on the force’s magnitude, the angle at which it is exerted, and the distance of the point of application from the axis of rotation. All other things being equal, the greater the force the greater the torque. A given force is maximally effective when its direction is perpendicular to the radial line directed from the axis of rotation to the point of application. The farther along this line a force is applied, at any angle other than parallel to the line, the greater the resulting angular acceleration. The first and third relationships imply that torque
is proportional to the magnitude of the force $|\vec{F}|$ and to the distance from the axis of rotation $|\vec{r}|$. The angular dependence behaves like a sine function, maximal at $\frac{\pi}{2}$ and minimal at 0. We might, as we’re at it, assign a direction to the torque: using the right-hand rule [see Figure 1.5], if the fingers curl in the direction of the resulting angular acceleration (not necessarily the motion, because a torque may slow down existing rotation), then the thumb gives the torque’s direction. This all sounds like a cross-product:

$$|\vec{\tau}| = |\vec{r} \times \vec{F}| = |\vec{r}| |\vec{F}| \sin \theta,$$

where $\theta$ is the angle between the radial line from the axis of rotation to the application point and the direction of the force.

Simple Example: We’ll compute the torque about a fixed point 0 on an object with mass number $M$ induced by the gravitational force (weight), assumed to be uniform over the entire volume of the object.

Imagine that the object is an assembly of constituents each with infinitesimal mass number $dm$, so that each constituent experience the gravitational force as weight $\vec{g} \, dm$. The differential torque, then, is $\vec{r} \times \vec{g} \, dm$, where $\vec{r}$ is the position of the constituent relative to the fixed point 0. To determine the torque, we sum the differential torques of each constituent:

$$\vec{\tau} = \int \vec{r} \times \vec{g} \, dm = \left( \int \vec{r} \, dm \right) \times \vec{g} = M \vec{r}_{CM} \times \vec{g} = \vec{r}_{CM} \times M \vec{g}.$$

The torque about a given point on an extended object due to the gravitational force is the object’s weight acting at the object’s center of mass.

10.3 Equilibrium

A rigid body is in equilibrium if a) its center of mass is moving uniformly ($\vec{a}_{CM} = \vec{0}$) and b) it is not rotating. This definition implies two general (six specific) conditions:

1. $\sum \vec{F}_{ext} = \vec{0}$, or, equivalently, $F_x \, net = F_y \, net = F_z \, net = 0$, and

2. $\sum \vec{\tau}_{ext} = \vec{0}$, or, equivalently, $\tau_x \, net = \tau_y \, net = \tau_z \, net = 0$.

When solving statics problems, it is most convenient, if possible, to arrange the reference frame so that all forces are coplanar. Then general condition 1 reduces to two specific conditions (forces in the plane chosen) and general condition 2 reduces to one specific condition (torques perpendicular to the chosen plane).
Chapter 11

Rigid Body Motion

11.1 Rotational Inertia

Just as the magnitude of an object’s rectilinear acceleration is proportional to the magnitude of the net external force exerted on the object, so too is the magnitude of an object’s angular acceleration proportional to the net external torque exerted on the object. [The direction of the rectilinear acceleration is the same as that of the force; the direction of the angular acceleration is the same as that of the torque.] For rectilinear dynamics, the proportionality constant is the object’s inertia, measured typically as a mass number, \( m \). The quantity in rotational dynamics analogous to inertia is rotational inertia, which is normally quantified as a moment of inertia, \( I \). [Note that this is a scalar, to be differentiated from the vector impulse, \( \vec{I} \).]

Rotational inertia depends on both the inertia and the shape of the object. If an object is composed of a number of discrete constituents, each with its own mass number \( m_1, m_2, \ldots, m_N \), and located respective distances \( r_1, r_2, \ldots, r_N \), from a given axis of rotation, then the object’s moment of inertia about a this axis is

\[
I = \sum_{i=1}^{N} m_i r_i^2 = m_1 r_1^2 + m_2 r_2^2 + \cdots + m_N r_N^2. \tag{11.1}
\]

For an object whose constituents are continuously distributed and exceedingly numerous, the sum of Equation 11.1 goes over to an integral:

\[
I = \int_V r^2 \, dm, \tag{11.2}
\]

where \( V \) indicates an integration over the entire volume of the object.

Obviously, the dimensions of moment of inertia are \( ML^2 \).

11.1.1 Radius of Gyration

For some applications, expressing the rotational inertia in terms of a single length, called the radius of gyration, is convenient. The standard symbol for radius of gyra-
tion is \( k \), and it’s defined by:

\[
k = \sqrt{\frac{I}{M}}, \tag{11.3}
\]

where \( M \) is the mass number of a particle with a moment of inertia equal to the extended body under consideration. In other words, if \( I \) is the moment of inertia around an axis of an object with mass number \( M \), then there is some length \( k \), such that \( I = Mk^2 \), which is the moment of inertia of a particle with mass number \( M \) rotating a distance \( k \) around the same axis. This \( k \) is the radius of gyration.

### 11.1.2 Theorems on Moments of Inertia

**Decomposition Theorem:** A extended object composed of multiple parts, each with moment of inertia \( I_1, I_2, \ldots, I_N \), has a moment of inertia

\[
I = \sum_{i=1}^{N} I_i = I_1 + I_2 + \cdots + I_N. \tag{11.4}
\]

**Parallel-axis Theorem:** If the moment of inertia about an axis through an object’s center of mass is \( I_{CM} \), then the moment of inertia about an axis parallel to, but displaced a distance \( s \) from, the axis through the center of mass is

\[
I = I_{CM} + ms^2, \tag{11.5}
\]

where \( m \) is the object’s mass number.

**Perpendicular-axis Theorem:** Consider a flat object lying in the \( xy \)-plane. Assume the mutually perpendicular components of the object’s moment of inertia about axes parallel to the reference lines (two of which lie in the \( xy \)-plane) are \( I_x, I_y \), and \( I_z \). Then these components are related by:

\[
I_z = I_x + I_y. \tag{11.6}
\]

### 11.2 Angular Dynamics

There is a rotational analogue to Newton’s Second Law for rectilinear motion, \( \sum \vec{F}_\text{ext} = ma \). For a rigid body rotating about an axis fixed in direction through the body’s center of mass, or an axis fixed in an inertial frame,

\[
\sum \vec{\tau}_\text{ext} = I\vec{\alpha}, \tag{11.7}
\]

where \( \vec{\tau}_\text{ext} \), the vector sum of the torques arising from external forces acting on the body, \( I \), the body’s moment of inertia, and \( \vec{\alpha} \), the body’s angular acceleration, are all determined about the given axis. As in the rectilinear case, the rotational form of Newton’s Second Law necessarily holds only in an inertial frame. If a thin rod, for
11.3. KINETIC ENERGY

example, is held horizontally with respect to the ground and released so as to maintain
its horizontal orientation, there is no rotation $\vec{\alpha} = \vec{0}$, relative to an end of the rod even
though there exists a torque, due to the gravitational force, acting on the center of mass,$\vec{\tau}_{ext} \neq \vec{0}$. Notice, though, that if the axis through the center of mass, instead of through
an end, of the rod were chosen, $\vec{\tau}_{ext} = \vec{\alpha} = \vec{0}$.

11.3 Kinetic Energy

In analogy with the rectilinear form, the kinetic energy of a rigid body rotating about a
fixed axis is

$$K_{rot} = \frac{1}{2} I | \vec{\omega} |^2,$$

(11.8)

where $I$ is the body’s moment of inertia and $\vec{\omega}$ is its angular velocity, both of which are
calculated relative to the fixed axis.

Energy is a scalar, so the total energy of any system is the algebraic sum of the
various forms of energy found in the system. The total kinetic energy of a rigid body
in motion, in particular, is the sum of the kinetic energy associated with the rectilinear
motion of its center of mass, known as translational kinetic energy $K_{tran}$, and the
kinetic energy associated with its rotation about the center of mass $K_{rot}$. If a rigid
body moves rectilinearly in a plane perpendicular to the axis through its center of mass
about which it simultaneously rotates, then the total kinetic energy takes a particularly
simple form:

$$K_{tot} = K_{tran} + K_{rot} = \frac{1}{2} m | \vec{v} |_{CM}^2 + \frac{1}{2} I_{CM} | \vec{\omega} |^2.$$

11.3.1 Instantaneous Rotational Power

If a rigid body experiences a net external force that is exerted through an angle about
the axis of rotation, then the instantaneous rotational power, rate at which work is
done by the net external torque on the the body is given by

$$P_{rot} = \vec{\tau}_{ext} \cdot \vec{\omega}.$$  

(11.9)
Chapter 12

Angular Momentum

12.1 Angular Momentum

Recall that, in rectilinear motion, the momentum $\vec{p} = m\vec{v}$. We should expect that, if the analogies continue, to find a rotational, or angular momentum,

$$\vec{L} = I\vec{\omega}. \quad (12.1)$$

In fact, Equation 12.1 holds only when the direction of $\vec{L}$ is constrained to be parallel to the direction of $\vec{\omega}$, which is not usually the case.

To arrive at the general form of angular momentum, we recall Equation 7.2, $\vec{F}_{\text{net}} = \frac{d\vec{p}}{dt}$, and Equation 10.3, $\vec{\tau} = \vec{r} \times \vec{F}$. Considering that a net force changes linear momentum, we should expect that a net torque changes angular momentum:

$$\vec{\tau}_{\text{net}} = \frac{d\vec{L}}{dt}. \quad (12.2)$$

Since

$$\vec{\tau}_{\text{net}} = \vec{r} \times \vec{F}_{\text{net}} = \vec{r} \times \frac{d\vec{p}}{dt},$$

and so

$$\vec{L} = \vec{r} \times \vec{p} = \vec{r} \times m\vec{v}, \quad (12.3)$$

for a single particle at position $\vec{r}$ relative to a chosen origin, and $\vec{p}$ is the particle’s linear momentum, $m$ its mass number, and $\vec{v}$ its velocity.\(^1\) For an extended object,

$$\vec{L} = \int \vec{r} \times \vec{v} \, dm. \quad (12.4)$$

Equation 12.2 expresses the principal of angular momentum, which might be thought of as Newton’s Second Law for angular motion. Note that the torque and

\(^1\)Note that $\vec{\tau} = \vec{\omega} \times \vec{r}$ and $\vec{\omega} \times \vec{v} = \vec{0}$.\]
angular momentum must be measured relative to the same origin. The principal holds if this origin is either the center of mass or a fixed point in an inertial reference frame.

The dimensions of angular momentum are $ML^2T^{-1}$.

### 12.2 Conservation of Angular Momentum

Equation 12.2 allows us to state a conservation law for angular momentum: if $\vec{\tau}_{net} = 0$, the total momentum of the system $\vec{L}_{tot}$ is constant.
Chapter 13

Harmonic Motion

13.1 Simple Harmonic Motion

Consider an object with mass number $m$ in equilibrium at position $s_0$. If the object is displaced slightly to position $s_i$ and released, it tends to accelerate back in the direction of equilibrium. We can model such behavior with a system of a particle with mass number $m$ attached to a fixed spring. A restoring force, given by Equation 6.8,

$$F_{res} = -k\Delta s,$$

where $k$ is the spring constant and $\Delta s = s - s_0$, thus acts on the object and instigates oscillatory behavior about $s_0$, the equilibrium position of the particle at the free end of the spring. Such motion is known as simple harmonic motion.

Newton’s Second Law, $\vec{F}_{net} = ma$, in this case takes the form

$$m\ddot{s} = -k\Delta s.$$

If we arrange reference frame so that $s_0$ is at the origin, the equation becomes

$$m\ddot{s} = -ks.$$

Rearranging this equation and defining $|\vec{\omega}| \equiv \sqrt{\frac{k}{m}}$, the so-called natural frequency, we get the equation of motion of a particle with mass number $m$ undergoing simple harmonic motion at the end of a spring with spring constant $k$:

$$\ddot{s} = -|\vec{\omega}|^2 s.$$

(13.1)

Since simple harmonic motion continuously retraces its trajectory, it is very similar to uniform circular motion, in which the rate at which the motion occurs is given by $\vec{\omega}$, the angular velocity. In fact, the solution of the second-order differential equation, Eq. 13.1, gives the position of the oscillating particle as a sinusoidal function of time:

$$s = S \cos (|\vec{\omega}| t + \delta),$$

(13.2)

where $S$ is called the amplitude, or maximum displacement, and $\delta \equiv -|\vec{\omega}| t_i$ is the so-called phase constant, or location in the repeated cycle at the instant designated
\( t = 0: s = S \cos (-|\vec{\omega}| t_i) = S \cos \delta \). Note that in the case of a spring oscillator, the angular velocity is determined by characteristics of the system: the square root of the spring-constant-to-mass-number ratio.

As with uniform circular motion, we can define a period of the oscillation

\[
T \equiv \frac{2\pi}{|\vec{\omega}|} = \frac{2\pi}{\sqrt{\frac{k}{m}}},
\]

and its frequency

\[
f = \frac{1}{T} = \frac{|\vec{\omega}|}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}.
\]

### 13.2 Damped Harmonic Motion

Of course, in real life, macroscopic oscillators are subject to resistive forces which progressively degrade, or damp, the amplitude. The equation of motion can be written:

\[
\ddot{s} = -|\vec{\omega}|^2 s - \frac{b}{m} \dot{s},
\]

where \( b \) is a positive scalar called the damping factor. The negative sign in front of this term, which multiplies the object’s velocity, indicates that the damping always opposes the motion. The angular velocity \( \vec{\omega} \) in this case, too, is that of the undamped oscillator.

Equation 13.3 has a solution (as you can check by substitution)

\[
s = e^{-\frac{b}{2m}t} \left( Ae^{\sqrt{\left(\frac{b}{2m}\right)^2 - |\vec{\omega}|^2} t} + Be^{-\sqrt{\left(\frac{b}{2m}\right)^2 - |\vec{\omega}|^2} t} \right),
\]

where \( A \) and \( B \) are constants determined from initial conditions. The initial exponential ensures that the amplitude of the motion decreases exponentially with time, dying out as \( t \to \infty \).

The details of the motion depend on the magnitude of \( b \) and the direction and magnitude of the initial velocity \( \dot{s}_i \). If \( b < 2m |\vec{\omega}| = 2\sqrt{mk} \), the exponential sum in the parenthesis becomes a sinusoidal function with \( |\vec{\omega}_d| = \sqrt{|\vec{\omega}|^2 - \left(\frac{b}{2m}\right)^2} \), the frequency of damped oscillations which asymptotically approach zero amplitude. If, however, \( b \geq 2\sqrt{mk} \), no oscillations occur, and the exact trajectory depends on \( \dot{s}_i \); \( b = 2\sqrt{mk} \) is known as the critical damping value.

### 13.3 Simple Pendulum

Consider a simple pendulum, say, a small ball with mass number \( m \) at the end of a light string of length \( \ell \). Two forces act on the ball, the tension in the string \( \vec{T} \) and the weight \( \vec{W} = mg \). As there is no acceleration along the line of the string, we may write \( ma_{||} = 0 \), or, if \( \theta \) is the angle of the pendulum from vertical, \( |\vec{T}| = m |\vec{g}| \cos \theta \). However, there is acceleration in the direction perpendicular to the string:
\[ ma_\perp = m \ddot{s} = -m |\vec{g}| \sin \theta, \]
or, since \( a_\perp = \ell \ddot{\theta}, \)
\[ m \ell \ddot{\theta} = -m |\vec{g}| \sin \theta. \]

But if \( \theta \) remains very small, then \( \sin \theta \approx \theta \), and this last equation becomes
\[ \ddot{\theta} = -\frac{|\vec{g}| \ell}{\ell} \theta, \]
which has exactly the same form as Equation 13.1, but with \( |\vec{\omega}| = \sqrt{\frac{\ell}{m}} \) instead of \( |\vec{\omega}| = \sqrt{\frac{k}{m}} \). Thus, a simple pendulum sent into motion with a small initial amplitude will behave like a simple harmonic oscillator with period \( T = 2\pi \sqrt{\frac{\ell}{|\vec{g}|}} \).