Measurement
# Physics Quantities Reducible to 7 Dimensions

<table>
<thead>
<tr>
<th>Dimension</th>
<th>SI Unit</th>
<th>Symbol</th>
<th>Example</th>
</tr>
</thead>
<tbody>
<tr>
<td>Length</td>
<td>meter</td>
<td>m</td>
<td>Standard door height: ~2 m</td>
</tr>
<tr>
<td>Mass</td>
<td>kilogram</td>
<td>kg</td>
<td>Loaf of bread: ~0.5 kg</td>
</tr>
<tr>
<td>Time</td>
<td>second</td>
<td>s</td>
<td>Heart beat period (rest): ~1 s</td>
</tr>
<tr>
<td>Electric Current</td>
<td>ampere</td>
<td>A</td>
<td>Electric kettle: ~10 A</td>
</tr>
<tr>
<td>Temperature</td>
<td>kelvin</td>
<td>K</td>
<td>Human: ~ 310 K</td>
</tr>
<tr>
<td>Luminous Intensity</td>
<td>candela</td>
<td>cd</td>
<td>Candle: ~1 cd</td>
</tr>
<tr>
<td>Amount of Substance</td>
<td>mole</td>
<td>mol</td>
<td>Cup of water: ~14 mol H₂O</td>
</tr>
</tbody>
</table>
Derived Units

- Angle: radian \([\text{rad}] = \frac{\text{arc length}}{\text{radius}}\)
- Area: \([\text{m}^2]\)
- Volume: \([\text{m}^3]\)
- Velocity: \([\text{m/s}]\)
- Acceleration: \([\text{m/s}^2]\)
- Force: \([\text{N}] = [\text{kg} \cdot \text{m/s}^2]\)
- Energy: \([\text{J}] = [\text{kg} \cdot \text{m}^2/\text{s}^2]\)
- Power: \([\text{W}] = [\text{kg} \cdot \text{m}^2/\text{s}^3]\)
# Scientific Notation: Powers of 10

<table>
<thead>
<tr>
<th>Prefix</th>
<th>Symbol</th>
<th>Decimal</th>
<th>Power of 10</th>
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<td>Y</td>
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<td>10^{24}</td>
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<td>zetta</td>
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<tr>
<td>yocto</td>
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</table>
Measurement

Quantitative comparison using one or more instruments

- Scaled artificial devices
- Calibrated ultimately to international standard
- Limited by precision and accuracy
- Associated numbers allow “objects” to be compared

Results in a number (measurand), another two or three numbers (uncertainty; confidence level), and units.
Precision and Accuracy

• Precision
  - Quantified as relative uncertainty
    \[
    \text{relative uncertainty} = \left| \frac{\text{uncertainty}}{\text{measurand}} \right|
    \]
  - Limited by random variations

• Accuracy
  - Difficult to quantify unless “true” answer known
    • significance = \left| \frac{\text{difference}}{\text{uncertainty of difference}} \right|
    • relative difference = \left| \frac{\text{difference}}{\text{“right” answer}} \right|
  - Limited by systematic variations
Good Measurement

• Satisfies well-specified requirements
  – Measurand
  – Uncertainty/Tolerance
• Employs appropriate instruments and methods
• Is carefully undertaken and cross-checked
• Follows recognized procedures
• Is reproducible
Quantification

- No measurement is—or can be—exactly right
- Establish acceptance (tolerance) criteria
  Otherwise wasteful

- Determine uncertainty
- Report results correctly
Being Quantitative

- Estimate with numerical approximations
- Provide values, uncertainties, and units for every relevant quantity
- Display information in graphs and tables
Components of a Measurement

- A symbol representing the measured quantity
- Measurand
- Uncertainty of measurement
- Units (if any)
- Confidence interval
- Probability distribution function (PDF) used to estimate uncertainty and confidence interval

\[ Z = (\text{measurand} \pm \text{uncertainty}) \text{ units (CI\%, PDF)} \]
Example: The length is 20 centimeters plus or minus 1 centimeter, at the 95% confidence level, of a triangular probability distribution function

\[ \ell = (20 \pm 1) \text{ cm, 95\% CL, triangular PDF} \]
Uncertainty

• Not an error, which should be corrected if known
• Quantifies doubts about the quality of a measurement
• Two quantities:
  – An interval or width of the certainty margin
  – A confidence level: probability estimate of certainty that identical measurements by others will fall within the interval
• Probability distribution function(s) sampled
Why is Quantifying Uncertainty Important?

- Quality of measurement: relative uncertainty \( \frac{\text{uncertainty}}{\text{measurand}} \)
- Understand results
- Compare with other results
- Quality control/tolerance
Sources of Variation/Uncertainty

- Instrument
- Measurand
- Procedure
- Calibration
- Skill
- Sampling
- Environment
If size and sign of a source is known, correct. Otherwise, each source in principle contributes independently to the overall uncertainty, and its magnitude must be estimated.
These are NOT Uncertainties

- Mistakes and errors
- Tolerances
- Specifications
- Accuracies
- Statistics
Kinds of Uncertainty

• Random
  - Repeated measurements (continuously) varied
  - Average (mean) probably best measurand estimate
  - Quantified by standard uncertainty of the mean

• Systematic
  - Typically not obvious
  - Repeated measurements give no new information
  - Alternative measurements or calculations usually necessary to quantify
Summary of Random and Systematic

- Not accurate - large systematic error
  - Not precise - large random error

- Not accurate - large systematic error
  - Precise - small random error

- Accurate - small systematic error
  - Not precise - large random error

- Accurate - small systematic error
  - Precise - small random error
Two Types of Uncertainty

• Type A
  – Statistics (repeated measurements)

• Type B
  – Estimates using other information
    • Experience
    • Calibration certificates
    • Manufacturer specifications
    • Publications
    • Common sense
    • ...

Usually, both must be evaluated
Assessing Uncertainty

- Identify sources
- Estimate magnitude from each source
- Combine contributions from all sources

There are clear rules for assessing each uncertainty and for combining them
Estimating Uncertainty: 8 Main Steps

1) Determine what needs to be measured (and how) and what needs to be calculated

2) Make measurement(s)

3) Estimate every input uncertainty and express in similar terms (same units and confidence level)

4) Check for independence of each input; if not, determine adjustments
Estimating Uncertainty: 8 Main Steps

5) Estimate measurand; make necessary corrections, if any

6) Calculate the standard uncertainty

7) Express the uncertainty in terms of a coverage factor to set size of the uncertainty interval; state confidence level

8) State result and how measurand and uncertainty were determined
All uncertainties must be in the same units and the same confidence level before they can be properly combined.
Reducing Uncertainty

- Calibrate
- Correct
- Know your instrument(s)
- Repeat measurements (at least 3 total)
- Check calculations
- Graph results as you take them
- Keep an uncertainty budget
Good Practices

• Read manuals and follow instructions
• Practice measurement before recording
• Record measurements, calculations, and any other possibly relevant information
• Check results as you get them
Reporting a Measurement (Conventions)

- Precision (decimal places) of uncertainty determines precision of measurement
- Uncertainty contains only 1 or 2 significant figures
- Round \textit{up} uncertainties, not measurands
- Decimal places of measurand match the decimal places of uncertainty
- Round measurand to same number of decimal places as uncertainty.
Rounding Measurand

- When next digit is less than five, don’t change
  - 2.541 → 2.5 to one decimal place
- When next digit is greater than five, change
  - 2.453 → 2.5 to one decimal place
- When next digit is exactly 5, change if odd, don’t change if even
  - 3.450 → 3.4 to one decimal place
  - 3.550 → 3.6 to one decimal place
Before Measuring

● Plan Measurement
  - Define measurand (object, system, model)
  - Select the variables
  - Clarify the relationship
  - Determine ranges and increments
    - Coarse measurements over wide range, fine in regions of interest
  - Consider the precision
  - Construct a measurement program
Make any measurement at least three times
If Measurements Vary

- Take between 4 and 10 measurements
- Find the average \( \bar{x} = \frac{\sum_{i=1}^{N} x_i}{N} \)
- Find the standard deviation \( s = \sqrt{\frac{\sum_{i=1}^{N} (x_i - \bar{x})^2}{N-1}} \)
- Find the standard uncertainty of the mean \( u_{\bar{x}} = \frac{s}{\sqrt{N}} \)
Weighted Average

- Quantities with different uncertainties weighted by factors inversely proportional to their respective uncertainties squared:

\[
\bar{x} = \frac{\sum_{i=1}^{N} x_i / u_i^2}{\sum_{i=1}^{N} 1 / u_i^2}
\]

\[
u_{x} = \sqrt{\frac{1}{\sum_{i=1}^{N} 1 / u_i^2}}
\]
Fitting is Averaging in 2-Dimensions
Weighted Fit
Probability Distributions

- Triangular
- Uniform (Rectangular)
- Normal
- Also, bimodal, skewed, binomial, Poisson….
Triangular Probability Distribution Function

![Triangle Probability Distribution Function](image)

- $\frac{1}{3\sigma}$
- $\bar{x} - \frac{a}{\sqrt{6}}$
- $\bar{x}$
- $\bar{x} + \frac{a}{\sqrt{6}}$
- $\bar{x} + a$

65%
Uniform (Rectangular) Probability Distribution Function

![Uniform (Rectangular) Probability Distribution Function](image-url)
Normal (Gaussian) Probability Distribution Function

99.7% of the data are within 3 standard deviations of the mean
95% within 2 standard deviations
68% within 1 standard deviation

By Dan Kernler - Own work, CC BY-SA 4.0, https://commons.wikimedia.org/w/index.php?curid=36506025
Converting PDFs for Combining

- Rectangular → Triangular: \( \frac{0.65}{0.58} u_{\text{rect}} \approx 1.12u_{\text{rect}} \)
- Rectangular → Normal: \( \frac{0.68}{0.58} u_{\text{rect}} \approx 1.17u_{\text{rect}} \)
- Triangular → Rectangular: \( \approx 0.86u_{\text{triangle}} \)
- Triangular → Normal: \( \approx 1.06u_{\text{triangle}} \)
- Normal → Rectangular: \( \approx 0.81u_{\text{norm}} \)
- Normal → Triangular: \( \approx 0.93u_{\text{norm}} \)
Standard Uncertainty $u$

- **Uncertainty of the measurand**
  - Standard deviation ($s$): spread of the distribution

- **Type A, $u_A$**
  - Mean, $\bar{x}$; standard deviation, $s$
  - Standard uncertainty of the mean, $u_A = s_{\bar{x}} = \frac{s}{\sqrt{N}}$

- **Type B, $u_B = \sqrt{u_A^2 + u_b^2 + u_c^2 + \cdots}$**

- **Combined standard uncertainty,**
  \[
  u_T = u_C = \sqrt{u_A^2 + u_B^2}
  \]
Type A Uncertainties

• Standard Deviation, $s$
  - Uncertainty of individual measurements
  - $\bar{x} \pm s \Rightarrow 68\%$ chance that a single measurement will be within one standard deviation of the mean

• Standard Uncertainty of the Mean, $s_{\bar{x}}$
  - Uncertainty of the estimate of the “true” value
  - $\bar{x} \pm s_{\bar{x}} \Rightarrow 68\%$ chance that the true value is within one standard uncertainty of the mean
Uncertainty of the Standard Deviation

\[ s_s = \frac{s}{\sqrt{2(N - 1)}} \]

- Informs precision (number of decimal places) of the standard deviation
Correlated Uncertainties
Are Uncertainties Inter-Related?

• Does a change in one uncertainty affect the magnitude of another uncertainty?
• If so, combining uncertainties is more complicated
• Plot one uncertainty against another from different measurements?
• Often difficult to determine, otherwise
Combined Standard Uncertainty

General Case

- Propagation of uncertainty
  - Combined measurands
  - Covariance

\[
\begin{align*}
    u_T^2 &= u_C^2 = \sum_{n=1}^{N} \left( \frac{\partial f}{\partial x_n} \right)^2 u_n^2 + 2 \sum_{n=1}^{N-1} \sum_{m=n+1}^{N} \frac{\partial f}{\partial x_n} \frac{\partial f}{\partial x_m} u_{nm} \\
    u_{nm} &= \frac{1}{N(N-1)u_n u_m} \sum_{k=1}^{N} (\delta x_k)_n (\delta x_k)_m
\end{align*}
\]

Correlation Coefficient
Combined Standard Uncertainty

Independent Case

• Propagation of uncertainty
  – Combined measurands
  – Covariance = 0

\[ u_T^2 = u_C^2 = \sum_{n=1}^{N} \left( \frac{\partial f}{\partial x_n} \right)^2 u_n^2 \]
Relative Uncertainty

\[ \frac{u_c}{\text{measurand}} \]

Combined relative uncertainty can't be smaller than

\[ \frac{u_i}{\text{measurand}} \]

Relative uncertainty of any contribution to \( u_c \)
Confidence Interval

• Determine approximate level of confidence such that

\[ x - ku_c < X < x + ku_c \]

\( k \) Coverage
Coverage Factor, $k$

- Uncertainties scaled consistently before combining: combined standard uncertainty
- Combined standard uncertainty scaled by a coverage factor: expanded standard uncertainty
- Typical confidence levels
  - One standard uncertainty: 58%, 65%, 68%
  - Two or three or five standard uncertainties
    - 90%
    - 95%
    - 99%
Components of a Measurement

• A symbol representing the measured quantity
• Measurand
• Uncertainty of measurement
• Units (if any)
• Confidence interval
• Probability distribution function (PDF) used to estimate uncertainty and confidence interval

\[ Z = (\text{measurand} \pm \text{uncertainty}) \text{ units (CI\%, PDF)} \]
Note: PDF vs Coverage Factor

• Some recommend stating the coverage factor rather than the PDF. These are equivalent given the confidence interval.
## Uncertainty Budget

<table>
<thead>
<tr>
<th>Source</th>
<th>PDF</th>
<th>Value</th>
<th>Scale Factor</th>
<th>$u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Resolution</td>
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<tr>
<td>Calibration</td>
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<tr>
<td>Instrument</td>
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</tr>
<tr>
<td>Uncertainty of mean</td>
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<table>
<thead>
<tr>
<th>Combined Standard Uncertainty</th>
<th>$u_T = u_C$</th>
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</table>

<table>
<thead>
<tr>
<th>Expanded Standard Uncertainty</th>
<th>$k=?$</th>
<th>$U$</th>
</tr>
</thead>
</table>
Making Inferences from Data

- Comparisons
- Determining the Frequency Distribution
  - Maximum Likelihood
- Determining the relationship between variables
  - Least Squares
  - Goodness of fit
Comparisons

- **Percent Difference**

  No

- **t-test**

  \[ t_{\text{difference}} = \frac{|x_2 - x_1|}{\sqrt{u_1^2 + u_2^2}} \]

  Yes
Comparisons

- Chi-Square

\[ \chi^2 \equiv \sum_{i=1}^{N} \frac{(y_i - x_i)^2}{u_{y_i}^2 + u_{x_i}^2} \]

Yes
Compliance

specified upper limit

specified lower limit

(a)  (b)  (c)  (d)