Astronomy 328 Homework #3: Solutions

1. Can gravitational energy supply a star's luminosity? According to the virial theorem – which we will prove later on in the course – the total energy of an interacting system is equal to one-half its potential energy: \( E = W/2 \). Since total energy of a gas is the sum of potential energy and thermal (kinetic) energy, this also means that its thermal energy is equal to minus one-half its potential energy.

   a. A “star” with mass \( M \) and radius \( R \) has constant density \( \rho \). What is its gravitational potential energy, in terms of \( M \) and \( R \)?

   This is a problem you probably did solve in your intro physics classes. The star’s density is \( \rho = 3M/4\pi R^3 \) Envision a spherical shell within the star, with radius \( r \) and thickness \( dr \). It has mass \( dm = 4\pi r^2 \rho dr = \left(3Mr^2/R^3\right)dr \). Inside this shell lies a mass \( M(r) = 4\pi r^3 \rho/3 = Mr^3/R^3 \), so

   \[
   W = -\int \frac{GM(r)}{r} dm = -\int_0^R \frac{GM^2 3Mr^2}{R^6} dr = -3GM^2 \left[ \frac{r^5}{5} \right]_0^R = -\frac{3GM^2}{5} R.
   \]

   b. The star shrinks in radius at a small but constant rate \( dR/dt = -v \), where \( v \) is a positive number with the units of velocity. At what rate \( dW/dt \) does the star’s gravitational potential energy change?

   Remember the chain rule?

   \[
   \frac{dW}{dt} = \frac{dW}{dR} \cdot \frac{dR}{dt} = -\frac{3GM^2}{5} v.
   \]

   All the constants in this expression are positive, so the potential energy is decreasing (i.e. getting more and more negative).

   c. Normally the change in potential energy would be accompanied by a change in thermal energy. But suppose that the star stays at constant temperature (constant thermal energy) during this collapse, and radiates the energy away that would normally add to its thermal energy. Suppose also that this radiation is the only radiation the star emits. What is the star’s luminosity?

   According to the virial theorem, the thermal energy in equilibrium is \( U = -W/2 \), so normally the star’s thermal energy would change during the collapse according to \( dU/dt = -(1/2)dW/dt \). Instead, the star radiates energy at that rate:

   \[
   L = -\frac{1}{2} \frac{dW}{dt} = \frac{3GM^2}{10} v.
   \]

   d. Suppose that the Sun derived its luminosity in this fashion. At what speed would it need to be shrinking, to produce the presently-observed luminosity? How long would it continue to shine? How long would it take for the luminosity to double? (Express your answers in to the last two questions in years.) Do you think this process can be ruled out as the source of the Sun’s power?

   \[
   v = \frac{10 L_\odot R_\odot^2}{3 GM_\odot^2} = 2.33 \times 10^{-4} \text{ cm sec}^{-1}.
   \]
That doesn’t seem like much – it would be hard to measure the associated Doppler shift – but of course the sun is not infinitely large and this could only go on for

\[ \Delta t = \frac{R_\odot}{v} = 3 \times 10^{14} \text{ sec} \approx 10^7 \text{ years} \).

The luminosity would not be constant, either, because of the dependence of \( L \) on \( R \). It will have doubled by the time the radius had shrunk to \( R_\odot / \sqrt{2} \), which takes

\[ \Delta t_2 = \frac{R_\odot - R_\odot / \sqrt{2}}{v} = \left(3 \times 10^{14} \text{ sec}\right) \left(1 - \frac{1}{\sqrt{2}}\right) = 2.8 \times 10^6 \text{ years} \).

The Solar system is 4.5 billion years old, and life has existed on Earth for at least the last three billion years; it seems unlikely that the Sun’s luminosity could have changed very much over billions of years, let alone millions, so this process probably does not produce the Sun’s luminosity.

2. Consider two protons (\( q = 4.8 \times 10^{-10} \text{ esu, m} = 1.67 \times 10^{-24} \text{ gm} \)), separated by \( 10^{-13} \text{ cm} \).

a. Calculate the magnitude of the proton’s electric repulsive force and gravitational attractive force. Can gravity hold the protons together?

b. Calculate the electrostatic and gravitational potential energies. Compare these to the binding energy of the deuteron, \( 2.2 \text{ MeV (3.55} \times 10^{-6} \text{ erg)} \). This binding energy is the opposite of the strong-interaction potential energy, and its relatively large value indicates the predominance of the strong interaction on nuclear size scales.

The magnitude of the repulsive force exerted by the protons on each other is

\[ F_E = \frac{q^2}{r^2} = 2.3 \times 10^7 \text{ dynes} \),

and that of the attractive gravitational force is quite a bit less:

\[ F_G = \frac{Gm^2}{r^2} = 1.9 \times 10^{-29} \text{ dynes} \).

The potential energies are

\[ W_E = \frac{q^2}{r} = 2.3 \times 10^{-6} \text{ erg} \]

and

\[ W_G = -\frac{Gm^2}{r} = -1.8 \times 10^{-42} \text{ erg} \),

both smaller in absolute magnitude than the binding energy of the deuteron, which must therefore be contributed for the most part by a third interaction (the strong interaction).
3. Assume that the Sun was initially composed 70% by mass of hydrogen. How many hydrogen nuclei, \( N \), were there? What is the total nuclear energy supply, \( NE/4 \), where \( E = 0.03m_pc^2 \), if all of the hydrogen could be fused into helium? It turns out that only about 13% of the hydrogen in a solar-type star lies within the parts of the star hot enough to participate in fusion during the star’s stay on the main sequence. Under these assumptions, what will the Sun’s main sequence lifetime be?

The number of hydrogen nuclei in the sun is \( N_H = 0.7 M_\odot/m_p = 8.3 \times 10^{56} \), so the total energy supply is

\[
E = \frac{N_H}{4} \times 0.03m_pc^2 = 9.4 \times 10^{51} \text{ erg}.
\]

The sun can’t use up its entire hydrogen supply, however; it will only use an amount \( E_{\text{eff}} = 0.13E = 1.22 \times 10^{51} \text{ erg} \) before it starts to have trouble supporting itself. If it burns up the energy \( E_{\text{eff}} \) at a rate \( L_\odot \), this takes

\[
t = \frac{E_{\text{eff}}}{L_\odot} = 3.2 \times 10^{17} \text{ s} = 1.0 \times 10^{10} \text{ years}.
\]

So far, it’s only lived \( 0.45 \times 10^{10} \) years, so we have quite a ways to go.