Measurement Uncertainties

1 Introduction

We all intuitively know that no experimental measurement can be “perfect”. It is possible to make this idea quantitative. It can be stated this way: the result of an individual measurement of some quantity is the sum of the actual value and an “error”. We associate the size of the error with the particular method of measurement—some techniques have inherently smaller errors than others (e.g., a micrometer vs. a yardstick.)

An important part of understanding an experiment and reporting its results is being able to determine the measurement uncertainty. A practicing scientist or engineer needs to understand measurement uncertainties both for the interpretation of measurements made by others, and for design of future measurements.

To find an uncertainty, it is necessary to find the range in which an individual error is likely to lie. For example, if we measure the speed of light to be $3.01 \times 10^{10}$ cm/sec, and a study of the measurement system indicates that the individual errors are likely to lie between 2.98 and 3.04, we would quote the result as

$$c = 3.01 \pm 0.03 \times 10^{10} \text{ cm/sec}$$

The uncertainty is usually explicitly displayed when experimental results are graphed (see, for example, Figure 1, below.)

2 Random and Systematic Errors

Measurement errors fall into two categories: random and systematic. The random error in an individual measurement is not predictable in exact magnitude or sign. Moreover, the average of random errors over many repeated, independent measurements of the same quantity is zero. Because of this, the average of several measurements is generally closer to the actual value than any of the individual measurements. The uncertainty in the case of random errors should indicate the range of results which would be obtained from many measurements. This should be a characteristic of the measurement apparatus.

The other type of error is “systematic”. Here the magnitude and sign of the error is always the same. The measurement result is shifted, up or down, by the same amount, in repeated measurements, so averaging many measurements will not improve things. Typically a measurement has both random and systematic errors. Sometimes they can be

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1We will define this more exactly, using probability, later.
independently estimated and quoted; if this were the case for the speed of light measurement, above, the result might be quoted

\[ c = 3.01 \pm 0.03 \pm 0.01 \times 10^{10} \text{cm/sec} \]

Here the last term is the systematic uncertainty.

There is actually a third type of error: a mistake or blunder, i.e., an error which is caused by the person performing the measurement, not by the apparatus. These are prevented or caught with attention and some art. One of the functions of written reports of experiments is to show readers, by descriptions and internal consistency checks, that this type of error has (probably) been eliminated.

Systematic errors are usually unique to specific techniques, and so the discussion below will concentrate on general methods for estimating random errors.

(The term “human error” is rather ambiguous; it is not clear to a reader if it means “blunder”, or one of the other types of errors. Its use in reports should be avoided.)

3 The RMS Deviation

One way to specify the range of a set of values is the RMS deviation. This stands for “Root” “Mean” “Square” deviation. “Mean” in this context means the average. Given a set of N values of x, the average, or mean x, denoted by \( \overline{x} \), is

\[ \overline{x} = \frac{1}{N} \sum_{i=1}^{N} x_i \]  

For the same set of x’s we can also define the RMS deviation from the mean, denoted by \( \sigma \), through the equation

\[ \sigma = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (x_i - \overline{x})^2} \]  

The quantity \( \sigma \) is a measure of how dispersed are the values of x away from their average, \( \overline{x} \). If all x’s have the same value, then \( \sigma \) is zero. The RMS deviation is frequently called the “standard deviation”.\(^2\) Note that there are several other equivalent expressions\(^3\) for \( \sigma \):

\[ \sigma = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (x_i - \overline{x})^2} \]  

and

\[ \sigma = \sqrt{\frac{1}{N} \sum_{i=1}^{N} (x_i - \overline{x})^2} \]  

As an example, consider the two sets of numbers in the following table. They might be measurements of the same quantity done with two different devices.

\(^2\)Strictly speaking, \( \sigma \) is the standard deviation if the x’s come from a particular kind of distribution called the “normal” or “Gaussian” distribution.

\(^3\)Sometimes \((N - 1)\) is written instead of \(N\). This is used when the value of \(N\) is rather small, less than 10, say. Then it turns out that the estimate of \( \sigma \) obtained with \((N - 1)\) is slightly closer to the limiting, large-N value.
Table 1

<table>
<thead>
<tr>
<th>Device</th>
<th>1</th>
<th>2.80</th>
<th>2.54</th>
<th>3.40</th>
<th>2.94</th>
<th>2.74</th>
<th>3.28</th>
<th>2.88</th>
<th>3.42</th>
<th>2.00</th>
<th>0.328</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>2.90</td>
<td>2.77</td>
<td>3.20</td>
<td>2.97</td>
<td>2.87</td>
<td>3.14</td>
<td>2.94</td>
<td>3.21</td>
<td>3.00</td>
<td>0.163</td>
<td></td>
</tr>
</tbody>
</table>

These values are plotted here:

![Device 1 Graph]

![Device 2 Graph]

The average value for both sets is the same, but the RMS deviation, \( \sigma \), is smaller for the second device. We would say that the second device makes measurements with greater precision than the first. If the number of measurements, \( N \), is increased, the values of \( \bar{x} \) and \( \sigma \) tend toward limiting values which are independent of \( N \). The value of \( \bar{x} \) is determined by the physical quantity being measured; the value of \( \sigma \) is determined by the measurement apparatus. It is common practice to use \( \sigma \) as the experimental uncertainty.

A more detailed study of errors involves consideration of the probability of obtaining a particular error in a measurement. One of the useful results of such a study is the following chart which gives the odds that an individual error will deviate from the mean by a specified number of standard deviations, \( \sigma \), for the “normal” error distribution.

<table>
<thead>
<tr>
<th>Deviation</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>1 ( \sigma )</td>
<td>.317</td>
</tr>
<tr>
<td>2 ( \sigma )</td>
<td>.046</td>
</tr>
<tr>
<td>3 ( \sigma )</td>
<td>.003</td>
</tr>
<tr>
<td>4 ( \sigma )</td>
<td>( 6 \times 10^{-5} )</td>
</tr>
<tr>
<td>5 ( \sigma )</td>
<td>( 6 \times 10^{-7} )</td>
</tr>
</tbody>
</table>
So it is very unlikely for an error to exceed 2-3 $\sigma$, but not so rare to exceed 1 $\sigma$. These results are useful in comparing different measurements. What matters in determining whether two measurements agree or disagree is not the absolute amount by which they differ, but the number of $\sigma$’s by which they differ. The importance of this is illustrated in the following figure, which displays several hypothetical measurements of the same quantity, together with a predicted value (from some theory, say.) The quantity is plotted along the vertical direction.

Figure 1

Experiment A deviates from the prediction by slightly more than 1 $\sigma$. It cannot be said to disagree, significantly, with the prediction. It is also evident that A and B agree with each other. Experiment B, however, deviates by over 3.5 $\sigma$ from the prediction. This is a significant disagreement, even though B is closer to the prediction than A! There are various measures of how close two quantities are to each other. The fractional deviation or the percentage deviation are examples. Sometimes the degree of closeness is interesting and important. But it cannot be used to find out if the two quantities are in agreement. For this, what matters is not the absolute difference, but the difference in units of uncertainty. So a calculation of the percentage deviation between two measurements, or between a measurement and a “book” value, yields a correct but not-very-useful number, unless the uncertainties are taken into account.

4 Error Estimation in Individual Measurements

As described above, one way to estimate the $\sigma$ of a certain measurement process is to make repeated, independent measurements and compute the RMS deviation from the mean. In certain cases, simpler methods may be used. A common source of random error is the
reading of a dial or scale. For a length measurement with a ruler, a good estimate of $\sigma$ is one half the smallest scale division. For measuring devices with digital readout, repeated measurements of the same quantity would all give the same result. An example is the resistance of a resistor, measured with a digital ohmmeter. The deviation of the actual resistance from the measured one is generally always the same. But the actual resistance (neglecting systematic errors) is somewhere within the interval defined by the smallest scale division. For many measurements of similar but not identical resistors, the error distribution has a box-like shape. In this case, the appropriate estimate of $\sigma$ can be shown to be $1/\sqrt{12} \approx .3$ times the smallest scale division. (Because the error distribution is not normal, the interpretation of deviations via the above table of probabilities, obtained from the normal distribution, is no longer valid.) But this $\sigma$ is still useful for least-squares fitting.

5 Error Propagation

Often it is necessary to combine measurements of different quantities, and to find the overall uncertainty. For example, if we measure a distance moved, $x$, during a time, $t$, for an object initially at rest with a constant applied force, then the acceleration is

$$a = \frac{2x}{t^2}$$

If we know the uncertainties $\sigma_x$ and $\sigma_t$, what is the uncertainty in $a$, $\sigma_a$? The general form of this question is: given a function

$$z = f(x, y)$$

what is $\sigma_z$, given $\sigma_x$ and $\sigma_y$? The answer can be shown to be, for the case of independent measurements of $x$ and $y$,

$$\sigma_z^2 = \left[\frac{\partial f}{\partial x}\right]^2 \sigma_x^2 + \left[\frac{\partial f}{\partial y}\right]^2 \sigma_y^2$$

For some common special cases, the above equation gives the following specific results:

1. The quantity $z$ is the sum or difference of $x$ and $y$. Then

$$\sigma_z^2 = \sigma_x^2 + \sigma_y^2$$

2. The quantity $z$ is the product or quotient of $x$ and $y$. Then

$$\left[\frac{\sigma_z}{z}\right]^2 = \left[\frac{\sigma_x}{x}\right]^2 + \left[\frac{\sigma_y}{y}\right]^2$$

3. The quantity $z$ is the average of $x$ and $y$, and $\sigma_x = \sigma_y$. Then since $\partial z / \partial x = 1/2 = \partial z / \partial x$,

$$\sigma_z = \frac{\sigma_x}{\sqrt{2}}$$
4. The quantity $z$ is the average of $N$ measurements, all with the same $\sigma$. Then the above result is easily generalized to

$$\sigma_{\text{avg}} = \frac{\sigma}{\sqrt{N}}$$

6 Problems

1. A certain device has made 7 measurements of some quantity, $x$: 3.03, 2.95, 3.10, 2.98, 3.05, 3.01, and 3.00. Compute $\sigma_x$.

2. An experimental measurement has a result $4.95 \pm .01$, and a predicted value of 5.00. What is the percentage deviation between the two? Do the two agree?

3. An experimental measurement has a result $4.0 \pm 1.1$, and a predicted value of 5.00. What is the percentage deviation between the two? Do the two agree?

4. Assuming $x$, $t$, and $a$ are related as in Equation 5, find an expression for $\sigma_a$ in terms of $\sigma_x$ and $\sigma_t$.

5. Length A is measured to be $7.5 \pm 0.3$ and length B is $7.0 \pm 0.6$. What is the difference between the two lengths, $L_B - L_A$, and the uncertainty in the difference? If the uncertainty in length A is reduced from .3 to .1, a factor of 3 improvement, what is the new uncertainty in the difference $L_B - L_A$?

6. Derive Equation 4, using the previous equations.

7 APPENDIX: Derivation of Error Propagation Equation

In the analysis of experimental data, we frequently combine measurements of different quantities. For simplicity, consider the combination of two quantities, $x$ and $y$. We may represent the combination as

$$z = f(x, y)$$

If we know the uncertainties $\sigma_x$ and $\sigma_y$, what is the uncertainty, $\sigma_z$, in $z$? To find $\sigma_z$, first expand $f(x, y)$ in a 2–dimensional Taylor series about $\bar{x}$ and $\bar{y}$:

$$z = f(\bar{x}, \bar{y}) + (x - \bar{x}) \frac{\partial f}{\partial x} + (y - \bar{y}) \frac{\partial f}{\partial y} + \ldots$$

(7)

Now

$$\sigma_z^2 = (z - \bar{z})^2$$

(8)

and we may take

$$\bar{z} = f(\bar{x}, \bar{y})$$
If we insert Equation 7 for \( z \) into Equation 8, and use the above equation, we get

\[
\sigma_z^2 = \left[ (x - \bar{x}) \frac{\partial f}{\partial x} + (y - \bar{y}) \frac{\partial f}{\partial y} \right]^2
\]

Or, since \((x - \bar{x})^2 = \sigma_x^2\) etc.,

\[
\sigma_z^2 = \left[ \frac{\partial f}{\partial x} \right]^2 \sigma_x^2 + \left[ \frac{\partial f}{\partial y} \right]^2 \sigma_y^2 + 2 \frac{\partial f}{\partial x} \frac{\partial f}{\partial y} (x - \bar{x})(y - \bar{y})
\]

(9)

The quantity \((x - \bar{x})(y - \bar{y})\) is called the covariance, \(\sigma_{xy}^2\).

If we expand the quantity

\[
\sigma_{xy}^2 = (x - \bar{x})(y - \bar{y})
\]

we get

\[
\sigma_{xy}^2 = xy - \bar{x}y - \bar{y}x + \bar{xy}
\]

\[= \bar{xy} - \bar{x}\bar{y}\]

If \(x\) and \(y\) are independent, a deviation in \(x\) is unrelated to a deviation in \(y\), so

\[\bar{xy} = \bar{x}\bar{y}\]

and this means that \(\sigma_{xy}^2 = 0\), and so Equation 9 reduces to Equation 6.