Correlated disorder and propagating modes in the Frenkel-Kontorova model

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The incommensurate Frenkel-Kontorova model in its pinned phase is shown to be equivalent to a system with correlated disorder. This correlated disorder appears as dimer-type “defects” on appropriately decimated lattices describing the phonon modes of the system. As a consequence of the special resonance condition where the reflected waves from two sites of the dimer undergo destructive interference, the decimated lattices exhibit Bloch-type phonon modes for energies that can be tuned by varying the nonlinearity parameter of the system. In a generalized two-parameter model, where the strength and the smoothness of the potential can be controlled independently, our study provides strong evidence of localization in a discrete quasiperiodic potential with an infinite number of steps. This localization boundary interwines with the parameter region exhibiting the Bloch-type states. [S0163-1829(99)12113-1]

I. INTRODUCTION

The subject of localization in aperiodic systems has been in the forefront of research since Anderson published his classical paper.1 In recent years the physics of disordered systems with internal correlations has attracted a great deal of attention.2 This is due to a delocalization mechanism resulting in Bloch states in such systems. Their existence is opposite to the usual belief that all states are localized in one dimension in the presence of randomness. A simple example of a random system exhibiting delocalized states is the random dimer model in which the site energies for pairs of lattice sites are assigned one of the two values at random.2 Here delocalization results from the destructive interference of reflected waves from the two sites of the dimer. Detailed theoretical studies3 have shown a variety of extended defects possessing the reflection symmetry could be made to satisfy the zero reflectance condition for energy close to the Fermi energy by tuning a parameter. Recently, extended states were also found in other aperiodic systems such as the Thue-Morse lattices4 and the Fibonacci lattices.4

In this paper, we show the existence of extended states in an aperiodic system with two competing length scales. We investigate the phonon modes in the classical Frenkel-Kontorova (FK) model5 and its generalization. Here the incommensurability is included in a simple way: springs connecting a one-dimensional chain of balls tend to keep the balls on a periodic lattice while a sinusoidal potential with competing periodicity frustrates the balls. The model exhibits the pinning-depinning transition characterized by the disappearance of a zero-frequency mode. In the pinned phase, the linear excitations of the system are determined by an incommensurate potential which can be shown to be represented by an infinite series of step functions.6 This discontinuous potential is due to the clustering of balls in the pinned phase. We show that such an incommensurate system is equivalent to a system with dimer-type defects.

The equivalence between the FK phonon equation in the pinned phase and the dimer model is established using perturbation theory starting from the limit of infinite pinning, the so-called anti-integrable (AI) limit.7 Using rigorous recursive symbolic dynamics and appropriate decimation, we find propagating Bloch-type phonons in the pinned phase of the FK model for all orders of the perturbation theory. In the lowest order, our solution is identical to the one for the standard random dimer model.7 In the FK model, the propagating modes appear as steplike structures named stepons in our earlier studies8 where their existence remained mysterious. In this paper, we resolve this mystery by tracing their origin into a special resonance condition satisfied due to the existence of hidden dimer-type correlations in the model. The study of phonon modes of the FK model bears a close resemblance to the subject of quasiperiodic Schrödinger operators.9 Our study throws light on an important topic, namely localization in a quasiperiodic potential with an infinite number of steps. Our numerical results suggest the existence of localization in an extended model with such a potential. This is interesting in view of previous results where the absence of localization has been proven rigorously in quasiperiodic systems with finite number of potential levels.10

In Sec. II, we review the characteristics of the phonon modes in the pinned phase of the FK model. This section consists of three subsections: (A) describes the equilibrium ball configurations, (B) reviews our exact renormalization scheme for the phonon modes, and (C) describes the renormalization bifurcation diagram. In Sec. III, we introduce a two-parameter generalized version of the FK model and describe the phase diagram in the two-parameter space. In Sec. IV, we discuss the perturbation theory and show how the incommensurate potential decomposes into a step potential where the number of steps increases with the order of perturbation. The symbolic representation of the lattice is explained in Sec. V. In Sec. VI, using an appropriate decimation scheme, we show that in all orders of the perturbation theory, the lattice can be cast into a lattice with dimer and...
double-dimer defects. The perturbation theory is discussed in detail up to the third order and then the generality of the results is argued for all orders. The results are summarized in Sec. VIII. The Appendix contains a detailed recursive derivation of the symbolic dynamics related to the perturbation theory.

II. PHONONS OF THE SUPERCRITICAL FRENKEL-KONTOROVA MODEL

A. Equilibrium ball configurations

The FK model describes a system of balls connected by Hooke’s springs in a sinusoidal potential. The energy of a configuration with ball positions \( x_n \) is given by

\[
H = \sum_n \left[ \frac{1}{2} (x_{n+1} - x_n)^2 + \frac{K}{4\pi^2} \cos(2\pi x_n) \right].
\]

In addition to the nonlinearity parameter \( K \), an important parameter is the average spacing \( \gamma \) of the balls

\[
\gamma = \lim_{n \to \infty} \frac{x_n - x_0}{n}.
\]

The equilibrium positions of the balls \( x_n \) are given by the iterates of the standard map,

\[
x_{n+1} + x_{n-1} - 2x_n = -\frac{K}{2\pi} \sin(2\pi x_n).
\]

The average spacing \( \gamma \) is the rotation number of the standard map. The heart of the problem is the mismatch between \( \gamma \) and the periodicity of the cosine potential. For rational values of \( \gamma \), the locations of the balls are described by a periodic orbit. On the other hand, for irrational \( \gamma \), the iterates \( x_n \) are confined on an invariant circle of the standard map if \( K \) is small. The breakup of the invariant circle at the critical value \( K_c \) corresponds to the pinning transition in the FK model accompanied with the disappearance of the zero-frequency phonon mode. The phonon modes are linear excitations of the equilibrium configurations of the balls satisfying the equation

\[
\psi_{n+1} + \psi_{n-1} + \epsilon_n \psi_n = E \psi_n,
\]

where the eigenvalue \( E \) is related to the phonon frequency \( \omega \) as \( E = -\omega^2 + 2 \), and \( \epsilon_n = K \cos(2\pi x_n) \) is the on-site phonon potential.

The pinning transition also changes the character of the smooth Bloch-type phonon excitations found in the subcritical parameter region preceding the transition. In the region \( K < K_c \), the potential \( \epsilon_n \) is a smooth function of the effective phase \( \theta = n/\gamma + \phi \) where the brackets denote the fractional part. Here \( \phi \) is an arbitrary phase factor. On the other hand, for \( K > K_c \) the phonon potential is determined by an underlying invariant Cantor set (cantorus) of the standard map. In our earlier paper,\(^8\) we investigated the nature of phonon eigenfunctions beyond the pinning transition, namely in the supercritical regime \( K > K_c \). Below, we summarize the renormalization approach used to obtain the bifurcation diagram characterizing the generalized phonon eigenstate. For convenience, we choose \( \gamma = \sigma^2 \), where \( \sigma = \frac{1}{2}(1 - \sqrt{5}) \) is the golden ratio. However, our method can be implemented for any irrational \( \sigma \).

B. Renormalization scheme

The incommensurability of the lattice is incorporated into the renormalization method by decimating out all lattice sites except those labeled by the Fibonacci numbers \( F_m, F_{m+1} = F_m + F_{m-1} \) \( (F_0 = 0, F_1 = 1) \), starting from an arbitrary initial site \( n \). At the \( m \)th step, the decimated phonon equation can be written as

\[
f_m(n) \psi(n + F_{m+1}) = \psi(n + F_m) + \epsilon_m(n) \psi(n).
\]

Using the additive property of the Fibonacci numbers, straightforward manipulations give the following exact recursion relations for the decimation functions \( \epsilon_m \) and \( f_m \):

\[
\epsilon_m(n) = -\frac{A_m(n) \epsilon_m(n)}{1 + A_m(n) f_m(n)},
\]

\[
f_m(n) = \frac{f_m(n + F_m) f_m(n + F_m)}{1 + A_m(n) f_m(n)}.
\]

These recursion relations can be iterated numerically for an arbitrary number of decimation steps provided the phonon frequency \( \omega \) and the sequence \( x_n \) describing the locations of balls can be calculated with arbitrary precision. As explained in our previous papers,\(^8,12\) the self-similarity of an eigenfunction can be captured by a nontrivial limit cycle for the decimation functions \( \epsilon_m \) and \( f_m \). The extended eigenfunctions lead to an asymptotic trivial fixed point whereas both decimations functions decay to zero in the case of a localized eigenfunction. Therefore, by studying the variation in the renormalization attractor as the parameter \( K \) varies between the depinned and pinned phase, we can infer the transport characteristics of the phonons.

C. Critical phonons and degeneracy points

Figure 1 shows for \( E_{\text{max}} \) (minimal phonon frequency) the bifurcation diagram of the renormalization flow based on the decimation function \( f_m \) as the nonlinearity parameter \( K \) is varied starting from the AI limit \( 1/K = 0 \) where the springs in the model disappear. The trivial limit cycle of the AI limit evolves into a nontrivial six cycle. This reflects the fact that fractal self-similar phonon eigenfunctions exhibit a period-six cycle on a lattice consisting of Fibonacci sites only. The ‘‘amplitude’’ of this limit cycle varies continuously in the pinned phase. The transition to pinning is signaled by a universal period-three limit cycle. It should be noted that the phonon modes never become localized in the pinned phase.

A rather intriguing aspect of the pinned phase is the existence of an infinite set of parameter values where the nontrivial limit cycle degenerates to a trivial fixed point. These degeneracy points\(^9\) appear to accumulate at the critical point corresponding to the pinning-depinning transition.

Figure 2 shows the nature of eigenfunctions at these degeneracy points. Comparing Figs. 2(a) and 2(b) we notice
that these modes follow the discontinuities of the potential and hence are latticelike. Such latticelike modes will be referred to as stepon modes.

In our analysis below, we show that the stepon modes of the type in Fig. 2 are extended states on a decimated lattice with Bloch index \( k = 0 \). In general, Bloch waves with other values of \( k \) can be obtained by tuning a parameter. The existence of Bloch waves on a renormalized lattice is due to the hidden dimer-type defects.

III. PHASE DIAGRAM OF THE GENERALIZED PHONON EQUATION

We study a generalized version of the FK phonon equation with two independent parameters \( K \) and \( \lambda \) so that in Eq. (4) \( \epsilon_n = \lambda \cos(2\pi n) \). The model reduces to the FK phonon equation when the two parameters are equal. This extension of parameter space is analogous to the analytic continuation into complex plane and will turn out to be crucial in understanding the stepon modes. In the two-parameter model, the parameter \( \lambda \) controls the strength while \( K \) determines the smoothness of the onsite "phonon" potential as the underlying invariant circle of the standard map undergoes the transition by breaking of analyticity at the critical value \( K = K_c \). In the region \( K < K_c \), the model falls into the universality class of the famous Harper equation which exhibits a transition from extended to localized states. Therefore, the two-parameter model provides one with a class of quasiperiodic systems that interpolates nicely between the Harper equation (\( K = 0 \)) and the FK phonon equation. As seen later, this important feature helps in understanding the absence of Anderson localization for the FK phonons.

Figure 3 shows the phase diagram of the model obtained using an exact renormalization scheme described in the previous section. The model exhibits an extended phase, a localized phase as well as a critical phase with self-similar eigenfunctions. In this two-parameter model, the degeneracy points become degeneracy curves. An interesting feature is the emergence of infinitely many of these curves from the "corner" \( 1/\lambda = 1/K = 0 \) along which the eigenfunctions are represented by an infinite series of step functions of the effective phase \( \theta \) (see Fig. 2). All except the right most curve intersect the standard FK limit \( K = \lambda \). The degeneracy curves accumulate at the critical parameter value \( K = K_c \) for the circle-cantorus transition of the standard map. As shown below, perturbation theory traces the origin of these stepon
modes to renormalized Bloch waves on appropriately decimated lattice.

Another richness of the two-parameter model is the existence of an Anderson-localized phase. This numerical result shows that a quasiperiodic system with infinite number of potential levels exhibit a phase transition from criticality to exponential localization. This is particularly interesting in view of the previous results where the absence of localization has been proven rigorously in quasiperiodic systems with finite number of potential levels. Therefore, the key feature of the phonon equation that results in localization is the fact that in the supercritical phase, the phonon modes are subjected to a discrete potential with infinite number of steps. As we discuss below, Anderson localization is not seen in any finite order perturbation theory, a result which is consistent with the exact result as the phonon potential for every finite order has a finite number of levels. However, our finite order perturbation theory describes the complexity of the phase diagram as the number of potential levels increase.

As seen from the phase diagram, the width of the localized phase shrinks to zero as the AI limit is approached. The localized phase appears above the FK limit and the localization boundary is not smooth. This suggests that for a discontinuous potential the threshold for localization depends in a rather subtle way on the parameter λ which controls the step size in the potential. Furthermore, the localized modes reside inside the tongues of the degeneracy curves which intertwines with the localization boundary. This suggests that the stepon modes play a very special role in the localization phenomenon in models where the underlying potential is not smooth. However, the specifics of that role eludes us at present. The fact that the degeneracy curves in the phase diagram extend all the way to K → ∞ suggests a perturbative approach to understanding their origin.

**IV. PERTURBATION THEORY NEAR THE ANTI-INTEGRABLE LIMIT**

We develop a perturbation theory near the anti-integrable limit by expanding the equilibrium configurations of the balls xₙ with κ = 1/K as the expansion parameter. Below we summarize the features of the power-series representation of the phonon potential and show that the potential is given by a series of step functions of θ. Interestingly, the discontinuities of this potential are found to be related to the forward and backward iterates of the map θᵢ⁺1 = θᵢ + γ. Since the discontinuities of the step modes are those of the underlying potential, this approach provides an elegant way to understand the geometrical fractal structure underlying these excitations.

In the cantorus regime, the hull function X defined by xₙ = X(nγ + φ) is a convergent series of step functions. The expansion of the hull function reads

\[ X(θ) = X₀(θ) + X₁(θ)κ + X₂(θ)κ² + \cdots. \]

Here \( X₀(θ) = \frac{1}{2} + \text{Int}(θ) \) is the hull function at the anti-integrable limit. The rest of the “expansion hull functions” \( Xₙ \) are solved recursively by substituting the above expansion into Eq. (3) and requiring that the terms on both sides match order by order in κ. The first five recursion relations obtained in this way are (the repeating argument θ is omitted in below)

\[
\begin{align*}
X₁ &= D[X₀] , \\
X₂ &= D[X₁] , \\
X₃ &= D[X₂] + \frac{2π²}{3}X₁² , \\
X₄ &= D[X₃] + 2π²X₁X₂ , \\
X₅ &= D[X₄] + 2π²X₁X₂² + 2π²X₁²X₃ - \frac{2}{15}π⁴X₅².
\end{align*}
\]

where

\[ D[X(θ)] = X(θ + σ) + X(θ - σ) - 2X(θ). \]

For the analysis of the discontinuities, it suffices to recognize the pattern in which the general \( Xₙ \) is given by \( D[Xₙ₋₁] \) and possible other terms containing other previous expansion hull functions. This implies that if \( Xₙ₋₁(θ) \) has a discontinuity at θ = θᵣ, then \( Xₙ(θ) \) has a discontinuity not only at θ = θᵣ, but also at θ = θᵣ ± σ. The discontinuities of \( X₀(θ) \) are located at the integer values of θ. As all the other \((> 0)\) expansion hull functions satisfy \( Xₙ(θ + 1) = Xₙ(θ) \), θ can be restricted to the interval \([0,1)\) so that the discontinuities of \( Xₙ \) are obtained as the first forward and backward iterates of the map \( θᵢ = θᵢ₋₁ + γ \) (mod 1) with \( θ₀ = 0 \). In other words, \( Xₙ \) has all the discontinuities of the previous expansion hull functions and two additional ones.

Using the expansion for the hull function, \( \cos[2πX(θ)] \) can be written as (again the repeating argument is omitted)

\[
\cos[2πX] = -1 + 2π²X₁²κ² + 4π²X₁X₂κ³ + \left[ 4π²X₁X₃ + 2π²X₂² - \frac{2}{3}π⁴X₁³ \right] κ⁴ + \cdots.
\]

It is clear from this expansion that the discontinuities of the expansion hull functions give rise to discontinuities of the potential. However, it is more difficult to determine all the discontinuities of the “approximate” potential obtained by truncating the expansion at some order \( κₙ \). In general, the term of the order \( κ'(l > 2) \) can be written as \([4π²X₁Xₙ₋₁ + cₙ(X₁, \ldots, Xₙ₋₂)]ₜₙ \). From this form we see that a discontinuity \( θ₂ᵣ \) of \( Xₙ₋₁(θ) \) gives rise to a discontinuity of \( \cos[2πX(θ)] \) at the order \( l \) provided \( Xₙ(θ₂ᵣ) \neq 0 \). However, \( Xₙ(θ) = 0 \) when \( θ ∈ [σ², 1 - σ²] \). Thus, the discontinuities of \( Xₙ₋₁ \) in this interval do not appear in the potential of the order \( κₙ \) unless they appear in some of the terms \( cᵢ \) with \( l ≤ p \). As it seems very difficult to get hold of the terms \( cᵢ \), for general \( l \), we develop a symbolic representation of the potential of an arbitrary order \( p \) based on the discontinuities of \( Xₙ₋₁ \). This leads to some “ghost” discontinuities which do not correspond to real discontinuities of the truncated potential at the order \( κₙ \). However, this does not affect our argument for the extended states.
In the Appendix, we show that a recursive scheme can be used to generate the symbol sequence for all orders in the perturbation theory.

The perturbation expansion for the phonon potential for all orders $p$ can be systematically obtained. However, to obtain the expansion for higher orders may require the usage of some symbolic manipulation package like MATHEMATICA. This on-site potential is then labeled by the corresponding symbol. As an example, we give the on site potential corresponding to the fourth order perturbation:

$$
\epsilon_a = -\lambda,
$$

$$
\epsilon_b = -\lambda \left[ 1 - 2 \pi^2 \kappa^2 + 16 \pi^2 \kappa^3 - (52 \pi^2 + 2 \pi^4) \kappa^4 \right],
$$

$$
\epsilon_c = -\lambda \left[ 1 - 2 \pi^2 \kappa^2 + 12 \pi^2 \kappa^3 - (54 \pi^2 + 2 \pi^4) \kappa^4 \right],
$$

$$
\epsilon_d = -\lambda \left[ 1 - 2 \pi^2 \kappa^2 + 12 \pi^2 \kappa^3 - (58 \pi^2 + 2 \pi^4) \kappa^4 \right].
$$

These expressions clearly illustrate the splitting of the levels as $p$ increases.

In the next section, we will show that for all orders $p$, we obtain a general lattice representation consisting of a renormalized dimer and double-dimer defects.

VI. DECIMATION SCHEME: HIDDEN DIMER

As discussed in the Appendix, the key feature underlying the symbolic representation is that the lattice can be viewed as consisting of three types of blocks. We now justify this by considering the $p=2$ and $p=3$ cases and then discuss the general $p$ case.

For $p=2$, the blocks $bb$ and $bbbb$, respectively, represent the dimer and double-dimer defects on the $a$ lattice. For higher values of $p$, the same picture is obtained after appropriate decimation. In the Appendix, we explicitly state for all $p$ the sites which have to be decimated. These decimated sites are in fact the sites where the Bloch wave is attenuated. As a result, the renormalized lattice has unattenuated Bloch waves which undergo phase shifts at the defect sites.

It turns out that by tuning a single parameter, a resonance condition is obtained where the Bloch wave $e^{ikn}$ passes through the renormalized dimer unattenuated with just phase shifts. The sites which lead to attenuation of the Bloch wave will always be decimated. We will obtain the phase shifts and the constraints on the parameters at the resonance. Although the decimation has to be done separately for each order, the phase diagram for all $p$ can be obtained in terms of the renormalized parameters by considering lowest order cases. Different $p$ cases are distinguished from each other by different renormalized on site energies and coupling terms.

In our analysis below, we write the diagonal term of the phonon equation as $E_n = E - \epsilon_n$ and its corresponding renormalized form as $\tilde{E}_n$. The renormalized off-diagonal terms will be denoted as $\tilde{V}$. For the bare lattice, all the off-diagonal terms are unity.
correlated disorder and propagating modes . . .

at the sites 0 and 1, the equations describing the system are

\[ a \text{-type lattice with a single dimer } b b \]

shown that the only solution is in fact identical to the one found in the standard

levels which generate the symbol sequence . . . b a b b b b b a b b b . . . By decimating the a sites out, one obtains the renormalized lattice . . . c c c b b c c b b . . . with dimer (c c c) and double-dimer (c c c c) "defects."

\[ \text{A. } p = 2 \]

Figure 5 (a) shows the locations of the balls in the two-level potential corresponding to \( p = 2 \). We first consider the a-type lattice with a single dimer \( b b \). If the dimer is placed at the sites 0 and 1, the equations describing the system are

\[ \psi_{-1} + \psi_1 - E_b \psi_0 = 0, \]

\[ \psi_0 + \psi_2 - E_b \psi_1 = 0, \]

\[ \psi_1 + \psi_3 - E_a \psi_2 = 0. \]

We look for a Bloch-wave solution where the plane wave \( e^{ikn} \) undergoes a phase shift \( \Omega_1 \) between the a- and b-type sites and a phase shift \( \Omega_2 \) between two b-type sites. It can be shown that the only solution is

\[ \Omega_1 = 0; \quad \Omega_2 = \pi - 2k, \]

\[ E_b = 0; \quad E_a = 2 \cos k, \]

\[ |e_b - e_a| = 2. \]

It is easy to see that the double dimer acts like two dimers: its four sites undergo phase shifts 0, \( \Omega_2 \), 0, and \( \Omega_2 \). This solution is in fact identical to the one found in the standard random dimer model. \(^2\)

\[ \text{B. } p = 3 \]

Unlike the \( p = 2 \) case, the picture of the lattice as consisting of dimer and double-dimer defects emerges only after decimating certain sites. We interpret the symbolic dynamics associated with this case [Fig. 5(b)] as the b lattice with defects \( c a c \) and \( c a c a c \). By decimating a, the lattice is transformed to a quasiperiodic lattice with dimers \( c c \) and double dimers \( c c c c \).

We first consider the single \( c a c \) defect. We decimate the site a and the resulting . . . b b c c b b . . . lattice where the dimer \( c c \) is at sites 0 and 1 is described by the following equations:

\[ \psi_{-2} + \psi_0 - E_b \psi_{-1} = 0, \]

\[ \psi_{-1} + \psi_1 - E_c \psi_0 = 0, \]

\[ \psi_0 + \psi_2 - E_c \psi_1 = 0, \]

\[ \psi_1 + \psi_3 - E_b \psi_2 = 0. \]

Therefore, the decimation of the site a results in the renormalization of the on site potential at the dimerized sites and also the renormalization of the coupling between these sites. The renormalized parameters are related to the bare parameters by the equations

\[ \bar{E}_c = \frac{E_c E_a - 1}{E_a}, \]

\[ \bar{V}_{cc} = \frac{1}{E_a}. \]

The decimated lattice can be shown to have a traveling wave where the Bloch wave \( e^{ikn} \) undergoes phase shifts \( \Omega_1 \) and \( \Omega_2 \).

\[ e^{i\Omega_1} = E_b e^{-ik} - e^{-2ik}, \]

\[ e^{i\Omega_2} = \frac{E_c - E_b e^{-ik} + 1}{V_{cc}}. \]

These equations for the phase shifts are valid provided the right hand side in both equations is of the unit modulus. Hence they determine the resonance conditions for the dimer.

It is easy to see that the above equations lead to two types of solutions:

\[ \Omega_1 = 0; \quad E_b = 2 \cos k, \]

and

\[ \Omega_1 = \pi - 2k; \quad E_b = 0. \]

The second solution exists for the infinite quasiperiodic chain only and cannot be realized for an isolated \( c a c \) defect.

It turns out that the defect \( c a c a c \) can be renormalized to a double dimer \( c c c c \) by decimating the a sites. The traveling wave at the four sites of this defect undergo phase shifts \( \Omega_1, \Omega_2, \Omega_1, \Omega_2 \), respectively. Hence, the defect acts like two dimers placed next to each other and does not cause any additional phase shifts. Therefore, the resonance condition for the existence of Bloch waves for the dimer defect is identical to that obtained for the double-dimer defect.

Figure 6 shows the regime for delocalization in the \( K - \lambda \) plane. Unlike Fig. 3 which shows the stepon modes with \( k = \Omega_1 = \Omega_2 = 0 \), the shaded part here shows the regime where the extended state exists by virtue of the resonance condition for other values of the Bloch index k. The boundaries of this region are determined by solutions with \( k = 0, \pi \) and \( \Omega_2 = 0, \pi \).

\[ \text{C. General } p \]

In order to discuss the general \( p \) case, we switch to the notation introduced in the Appendix where we show that the

\[ \psi_{-1} + \psi_1 - E_c \psi_0 = 0, \]

\[ \psi_0 + \psi_2 - E_c \psi_1 = 0, \]

\[ \psi_1 + \psi_3 - E_b \psi_2 = 0. \]
lattice can be divided into three types of blocks: the block $sSs$ (type I) or a single $s$ (type II), the dimer part $TuUuT$, and the double-dimer $TuUuTuUuT$. The capital letters describe blocks of symbols that can be decimated and hence the renormalized lattice consists of the dimers $uu$ on the $s$ lattice. Therefore, for type I, the renormalized equations with dimer $u_1u_2$ sandwiched between the ordered sites $s_1s_2$ and $s_3s_4$ can be written as

\[ \psi_{s_1} + \bar{V}_{su}\psi_{u_1} - \bar{E}_u\psi_{s_2} = 0, \]
\[ \bar{V}_{su}\psi_{s_2} + \bar{V}_{uu}\psi_{u_2} - \bar{E}_u\psi_{s_1} = 0, \]
\[ \bar{V}_{uu}\psi_{s_1} + \bar{V}_{su}\psi_{s_2} - \bar{E}_u\psi_{u_1} = 0, \]
\[ \bar{V}_{su}\psi_{u_2} + \psi_{s_3} - \bar{E}_s\psi_{s_4} = 0. \]

The physical consequence of this renormalization are the phase shifts $\Omega_1$ and $\Omega_2$ at the first and the second site of the renormalized dimer,

\[ e^{i\Omega_1} = \frac{\bar{E}_s e^{-ik} - e^{-2ik}}{\bar{V}_{su}}, \]
\[ e^{i\Omega_2} = \frac{(\bar{E}_u - \bar{E}_s) e^{-ik} + 1}{\bar{V}_{uu}}. \]

By demanding that the phase shifts $\Omega_1$ and $\Omega_2$ are real, we obtain two equations which give the resonance condition:

\[ \bar{E}_s^2 + 1 - \bar{V}_{su}^2 = 2\bar{E}_u\cos k, \]
\[ (\bar{E}_s - \bar{E}_u)^2 + 1 - \bar{V}_{uu}^2 = 2(\bar{E}_s - \bar{E}_u)\cos k. \]

Unlike the $p=3$ case, both $\Omega_1$ and $\Omega_2$ depend upon the parameters. Furthermore, the equation determining the energy does not decouple from the equation restricting the parameters.

As $p$ changes, the dependence of renormalized couplings and potentials on the bare parameters changes resulting in changes in the phase shifts. However, the general physical picture remains the same: At the resonance for a dimer placed at the sites 0,1, the Bloch wave for sites $(-2, \ldots, 3)$ takes the form $e^{-2ik}e^{-ik}e^{i\Omega_1}e^{ik+\Omega_1+\Omega_2}$, $e^{i(2k+2\Omega_1+\Omega_2)}$. The corresponding wave with double dimer at sites $(0, \ldots, 3)$ has the form $e^{-2ik}e^{-ik}e^{i\Omega_1}e^{i(k+\Omega_1+\Omega_2)}e^{i(2k+2\Omega_1+\Omega_2)}e^{i(3k+2\Omega_1+2\Omega_2)}$ for sites $(-2, \ldots, 5)$. There is a similar structure as in the type-II case.

For $p=4$, the renormalized parameters are related to the bare parameters by the equations

\[ \bar{V}_{su} = \frac{1}{\bar{E}_u}. \]
\[ \tilde{V}_{aa} = \frac{1}{(E_a^2 - 1)E_a}, \]  
\[ E_s = \frac{E_a E_a^2 - 1}{E_a}, \]  
\[ \tilde{E}_a = E_a - \frac{1}{E_a} \frac{E_b}{E_b - 1}. \]  

It should be noted that the degeneracy curves in Fig. 3 describe numerically obtained stepon modes with \( K = 0, \Omega_1 = \Omega_2 = 0 \). Therefore, we conclude that the delocalized modes exist also for the system obtained by taking into account the full expansion up to infinite order. In this case, the stepon eigenfunction is represented by an infinite series of step functions. Therefore, the stepon modes of the FK model can be thought of as Bloch waves on an infinitely many times renormalized lattice.

**VII. ASYMPTOTIC FORM OF THE DEGENERACY CURVES**

One of the important results of our analysis is the fact that it predicts the asymptotic form of the degeneracy curves shown in Fig. 3. Using the explicit values of the on-site energies results in the following relationship between the two parameters \( \kappa \) and \( \lambda \) for the three leftmost branches:

\[ \lambda = \frac{1}{4\pi^2} \kappa^{-3} + O(\kappa^{-2}), \]  
\[ \lambda = \frac{1}{4\pi^2} \kappa^{-4} - \frac{1}{4\pi} \kappa^{-3} + O(\kappa^{-2}), \]  
\[ \lambda = \frac{1}{4\pi^2} \kappa^{-4} + O(\kappa^{-2}). \]

Equation (48) was obtained using the \( p = 3 \) perturbation theory while the \( p = 4 \) theory gives all three solutions (48)–(50). We conjecture that the rest of the degeneracy curves can be explained by the higher order perturbation theory. Due to certain peculiarities of the expansion there are special values of \( p \) (e.g., \( p = 5 \)) for which \( \epsilon_a = \epsilon_b \) for the FK model. The corresponding order does not give rise to a new stepon solution but only to refinements of the previous solutions.

Furthermore, it should be noted that the degeneracy curve associated with the \( p = 2 \) solution is missing in Fig. 2 because the solution implies \( \lambda \sim -1/(\pi^2 \kappa^2) \) and hence appears for negative values of \( \lambda \) only.

**VIII. SUMMARY AND CONCLUSIONS**

Our numerical results along with the systematic perturbation and decimation theory demonstrate that the incommensurate FK model in the pinned phase is related to the class of models with correlated defects and exhibits unattenuated Bloch-type phonon modes on the renormalized lattice. The sites with attenuation can be decimated and the resulting lattice up to all orders in the perturbation theory consists of a lattice with dimer and double dimer defects. Therefore, our model is mapped to the random dimer model. Using a similar decimation scheme where sites with attenuation are decimated, the binary alloy model also gets mapped to the dimer model. We would also like to point out that our general solutions are valid for the type of defects we discuss irrespectively of whether they originate from quasiperiodic, chaotic or correlated random processes. At resonance, the whole complexity of the defect is absorbed in the phase shifts.

The existence of localized modes in the two-parameter model for \( \lambda \gg K \) is an important result. The potential seen by the phonons is purely discrete and takes an infinite number of distinct values in the cantorus phase. This allows the possibility of localized states in contrast with the earlier work where the absence of localization was proven for quasiperiodic potentials with a finite number of discrete levels. The intriguing fact is that the onset to localization depends in a rather subtle way on the size of discontinuities of the quasiperiodic potential. The absence of localization in the FK model is due to the fact that the nonlinear potential is not strong enough to localize the phonons.

The extended two-parameter model proposed here exhibits both correlated defects and the localization transition. This model also describes eigenstates of a one-dimensional Hamiltonian

\[ H = \sum \left[ a_{n+1}^\dagger a_n + \lambda \cos(2\pi x_n) a_{n+1}^\dagger a_n \right], \]

where the arguments of the sinusoidal potential \( x_n \) are the iterates of the standard map. In particular, the above model is the fermion representation of the quantum XY spin in a transverse magnetic field \( \lambda \cos(2\pi x) \). The existence of two spectral transitions, namely from extended to critical and from critical to localized states along with the cascade of resonance transitions due to the zero reflectance condition shows the richness and complexity of systems where the quasiperiodic potential has an infinite number of levels.

The propagating stepon modes seem to play a special role in determining the onset to Anderson localization. Proper understanding of how the stepon modes determine the localization boundary may be difficult as the perturbation theory in any finite order does not predict the existence of exponentially localized modes in the model.

The study of the two-parameter model helps one to understand an important distinction between the FK model and the Harper equation. First of all, the absence of localization in the FK model is due to the fact that the nonlinear potential is not strong enough to localize the phonons. The extended FK model, where the strength and the smoothness are controlled independently, does exhibit the localization transition for \( \lambda \gg K \). However, the localized phase for \( K > K_c \) exhibits an important distinction from that of the Harper equation. The localized eigenfunctions (once the exponentially decaying part is factorized out) for \( K < K_c \) exhibit universal self-similar fluctuations characterized by a unique strong-coupling renormalization fixed point. In contrast, these fractal fluctuations in the cantorus regime \( K > K_c \) appear irregular and defy any simple renormalization explanation.

Our study suggests that any deterministic potential (chaotic or otherwise) which results in clustering of the particles (thereby leading to a discontinuous potential) may exhibit (hidden) dimer-type defects. Therefore, the existence of
propagating phonon modes in the pinned FK model may be relevant to pinned flux and vortex lattices. Our perturbative analysis along with symbolic representation may be a useful tool in various applications of the FK model such as transient memories\textsuperscript{15} and friction\textsuperscript{16}. Furthermore, our RG scheme can be used to determine the parameter values satisfying the resonance condition for all energies as such points are characterized by the trivial renormalization fixed point.

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APPENDIX: SYMBOL SEQUENCES

In this Appendix, we derive the general form of the symbol sequence for all orders of the perturbation theory. The lattice can be viewed as composed of three types of blocks denoted by $O$, $D$, and $D_2$ representing the “ordered,” dimer, and double-dimer parts of the lattice, respectively. The basis for this decomposition is the fact that block $D$ does not introduce a phase shift while the $D$ and $D_2$ blocks introduce phase shifts in the Bloch waves.

It turns out that blocks $D$ and $D_2$ can be further decomposed as $D = TuTuT$ and $D_2 = TuTuTuTuTuT$, while block $O$ has two possible representations resulting in lattices of types I and II. For the type I, $O = sSs$ whereas for the type II, $O = s$. Furthermore, for the type II, $s = a$ corresponding to the central interval containing $\theta = 1/2$. Here the capital letters represent blocks of symbols of varying lengths depending on the order of perturbation. $s$ and $u$ are distinct symbols and $S$, $T$, and $U$ are blocks of other symbols than $s$ and $u$. These blocks are mutually exclusive, i.e., they do not contain any common symbols. Some of the blocks can be also empty with no symbols. In the following, we refer to an empty block by the symbol $\emptyset$.

The above forms for blocks $D$, $D_2$, and $O$ imply that a type-I symbol sequence has the form

\[ ...sSsTuUuTsSsTuUuTuUuTsSs... \]  \hspace{1cm} (A1)

and the corresponding form for a type-II sequence becomes

\[ ...sTuUuTsTuUuTuUuTs... \]  \hspace{1cm} (A2)

By decimating the blocks $S$, $T$ and $U$ out, one is left with dimer and double-dimer $u$ defects on the $s$ lattice for arbitrary $p$.

Table I shows the explicit forms of the blocks in the notation used in the previous sections. The complexity and order underlying the symbolic dynamics is clearly seen in this table. The needed mirror symmetry is explicit here as blocks $S$, $U$, and $T$ exhibit the reflection symmetry.

The schematic geometric picture underlying the symbol sequences of type I is shown in Fig. 7(a). The new discontinuities in $X_{p-1}(\theta)$ at the order $p$ are $\theta_+ = f^{p-1}(0)$ and $\theta_- = f^{-(p-1)}(0)$ where

\[ f(\theta) = \theta + \gamma \mod 1. \]  \hspace{1cm} (A3)

The discontinuity $\theta_+$ is the common border of two neighboring $\theta$ intervals denoted by $s_+$ and $u_+$. Their lengths are related by the golden mean so that $s_+$ corresponds to the shorter interval and $u_+$ to the longer interval. There is a similar symmetrical structure around $\theta_-$ with intervals labeled by $s_-$ and $u_-$. In terms of these intervals, the symbolic blocks take the forms $O = s_-S_s$, $D = Tu_+U_+T$, and $D_2 = Tu_+U_+Tu_+U_+T$ for the type-I cases $p = 3,4,6,...$. The above forms mean that the mapping $f$ takes the $s_-$ interval onto the $s_+$ interval through the set of other intervals forming the symbol sequence $S$. Furthermore, the $s_+$ and $u_+$ intervals are mapped onto the $s_-$ and $u_-$ intervals through the set of intervals with sequence $T$. Finally, $f$ maps the $u_-$ interval onto the $u_+$ interval through the intervals forming

![FIG. 7. Geometric pictures associated with symbol sequences of type I (a) and II (b).](image-url)
The geometric picture for the sequences of type II is different because in this case \( \theta_+ \) and \( \theta_- \) form the borders of the same central \( s \) interval. In other words, \( \theta_+ \) and \( \theta_- \) come closer to \( \theta = \frac{1}{2} \) than any other previous forward or backward iterate of \( f \) with \( \theta_0 = 0 \). Such \( p \) values (\( p = 2, 5, 18, \ldots \)) are obtained from \( p = G_m + 1 (m = 1, 2, \ldots) \) where \( G_{m+1} = 4G_m + G_{m-1} \) with \( G_0 = 0, G_1 = 1 \). The \( u_+ \) and \( u_- \) intervals can still be defined in the same way as before. In terms of these and the central \( s \) interval the symbolic blocks become \( O = s, D = T u_+ u_- T, \) and \( D_2 = T u_- u_+ T u_+ u_- T \). These forms mean that the \( s \) and \( u_+ \) intervals are mapped onto the \( s \) and \( u_- \) intervals through the set of intervals \( T \), while \( f \) takes the \( u_- \) interval onto the \( u_+ \) interval through the intervals forming the sequence \( U \). The corresponding schematic picture is shown in Fig. 7(b).

The proof of the above symbolic forms is recursive: Firstly, form (A1) is valid for \( p = 3 \) with \( S = T = \varnothing \) and \( U = a, s, b = b \) and \( u = c \) (refer to Sec. V). Secondly, it is possible to derive rules on how the blocks and symbols for \( p + 1 \) are obtained from those for \( p \) assuming the general form is valid for \( p \). It turns out that these rules depend primarily on the block \( T = t_1 t_2 \ldots t_n \). Four different cases can be shown to exist: (1) \( T = \varnothing \), (2) \( T = a \), (3) type I with nontrivial \( t_1 \neq a \), and (4) type II with nontrivial \( t_1 \neq a \). Each case has its own recursion rules which preserve the reflection symmetry of the blocks \( T, U, \) and \( S \). The reflection symmetry is trivially valid for \( p = 3 \).

We now present the necessary lemmas with recursion rules which together form the proof of the general form. Although the assumptions of the lemmas may first appear rather restrictive, later we show that a symbol sequence always satisfies one set of these assumptions (see Proposition).

**Lemma (Case 1):** If \( T = \varnothing \) and the symbol sequence is of the type I with \( O = s_1 s_2 s_3, D = u_0 u_1 u_2 u_3, \) and \( D_2 = u_0 u_1 u_2 u_3 \), then the next order has a symbolic form of the type I where the new symbols and blocks are obtained from the recursion relations
\[
s_+ = u_+ , \quad s_- = u_- , \quad u' = \text{new symbol} \qquad (A4)
\]
\[
S' = \varnothing , \quad T' = U , \quad U' = s_1 s_2 s_3 . \qquad (A5)
\]

**Proof:** Because \( T = \varnothing \) maps the \( s_+ \) and \( u_+ \) intervals directly onto the \( s_- \) and \( u_- \) intervals. As a consequence, \( f(\theta) \) divides the \( u_- \) interval into two subintervals \( u_{-1} \) and \( u_{-0} \) as shown in Fig. 8(a). Let us further define subintervals \( u_{+0} \) and \( u_{+1} \) which are symmetrical counterparts of \( u_{-1} \) and \( u_{-0} \). It is easy to check from the figure that these definitions are in harmony with the above forms for \( O, D, \) and \( D_2 \). Now it can be seen from Fig. 8(a) that the symbol sequence of the level \( p + 1 \) is of the type I with \( O' = s'_- s'_+ D' = T' u'_+ u'_- T' \), and \( D'_2 = T' u'_+ u'_+ T' u'_- T' \). Provided one takes the new symbols and blocks as given by the recursion rules. ⊥

**Example:** The symbol sequence for \( p = 3 \) is \( \ldots b c a c b c a c a c c b b \ldots \), i.e. \( s = b, u = c \) and \( U = a \).

Furthermore, it is easy to check that the geometric picture of Fig. 8(a) is valid for this order. Using the above rules we obtain for \( p = 4 \) the symbol sequence \( \ldots b d a d b d b d b d a c b b d b \ldots \) where \( d \) is the new symbol.

**Lemma (Case 2):** Assume \( T = t_1 \ldots t_n \) where \( n \geq 1, t_1 = t_n = a \), and the sequence is of type I. Then the new sequence is of type II with
\[
s' = t_1 , \quad u' = \text{new symbol} , \qquad (A6)
\]
\[
T' = u_- u_+ , \quad U' = s_1 s_2 s_3 . \qquad (A7)
\]

**Proof:** Because \( t_1 = t_n = a \), the image \( f(\theta) \) and the preimage \( f^{-1}(\theta) \) are both located in the central interval dividing into three subintervals with symbols \( u', s', \) and \( u' \). From Fig. 8(b) it is easy to see that the new symbol sequence \( p + 1 \) has the type-II form with \( O' = s'_- , D' = T' u'_+ u'_- T' \), and \( D'_2 = T' u'_+ u'_+ T' u'_- T' \) if the new symbols and blocks are taken according to the above recursion rules. ⊥

**Example:** Let us rewrite the sequence for \( p = 4 \): \( \ldots c a c b d b c d d a b d b d b d a c b b \ldots \) This has the regular form with \( s = c, u = d, T = a, \) and \( U = b b \).
The above rules lead to the sequence \( \ldots cc c e d b d b c e c d b d b c d b d b c d \ldots \) for \( p = 5 \) with new symbol \( e \).

**Lemma (Case 3):** Assume that \( T = t_1 \ldots t_n \) with \( t_1 \equiv t_n \) and the sequence is of the type I. If \( T \) has a nonvanishing first symbol \( t_1 \neq a \), then \( n \geq 2 \) and the next order is of the type I with new symbols and blocks

\[
s'_- = t_n, \quad s'_+ = t_1, \quad u' = \text{new symbol},
\]

\[
S' = s_s s'_+ t_n \ldots t_{n-1} .
\]

Here \( T' = \emptyset \) if \( n = 2 \).

**Proof:** The subintervals through the set intervals forming the sequence \( T = t_1 t_2 \ldots t_n \). This implies that \( f(\theta_+) \) divides the interval labeled by \( t_1 \) into two subintervals \( s'_+ \) and \( u'_+ \). In the same way, \( f^{-1}(\theta_-) \) divides the interval labeled by \( t_n \) into two subintervals [see Fig. 8(c)]. It is not possible that \( n = 1 \); otherwise \( f(\theta_+) \) and \( f^{-1}(\theta_+) = 1 - f(\theta_-) \) would fall into the same interval which could only be the central one with label \( t_1 = a \). Thus, \( t_1 \) and \( t_n \) correspond to two distinct intervals with identical symbols and the type-I sequence is obtained for the next order with the above recursion relations.

**Lemma (Case 4):** Assume that \( T = t_1 \ldots t_n \) with \( t_1 \equiv t_n \) and the sequence is of the type II. If \( T \) has a nonvanishing first symbol \( t_1 \neq a \), then \( n \geq 2 \) and the next order is of the type I with new symbols and blocks

\[
s'_- = t_n, \quad s'_+ = t_1, \quad u' = \text{new symbol},
\]

\[
S' = s_s t_1 t_n \ldots t_{n-1} .
\]

According to the proposition, any symbol sequence can be generated using the recursion rules which preserve the general form and the symmetry of symbol sequences. In Table I, we show the results of a few applications of these rules.

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