Pendulum:
\[ \ddot{\theta} + \omega_0^2 \sin \theta = 0 \rightarrow \text{Nonlinear equation} \]

Phase portrait: (1) Find equilibrium points.
\[ \dot{\theta} = 0 \Rightarrow \theta^* = 0 \text{ or } \theta^* = \pi \]

(2) Stability of the equilibrium point: linearize about the fixed points.
\[ \theta = \theta^* + \epsilon \]

\[ \text{Ellipse} \]

\[ \text{Hyperbola} \]
(1) Note: each curve or trajectory in phase space describes motion with fixed energy or initial condition.

(2) Motion is regular.

(3) If this pendulum is subjected to driving, like \( \ddot{\theta} + \omega^2 \sin \theta = A \sin \omega t \), the resulting motion may be very complicated.

(4) Compare with the linear motion:

(5) Two or higher dimensional nonlinear systems also could exhibit very complicated motion.
Dissipative Systems:

Compare linear dissipative & linear non-dissipative (called Hamiltonian) systems.

Note, there are NO attractors in the Hamiltonian systems as different initial conditions give different trajectories.

Note, all initial conditions settle on the same phase space trajectory as $t \to 0$.

Such trajectories (point or a closed curve) are called attractors.

Definition: Attractor: Final trajectory ($t \to \infty$) to which all initial conditions are attracted. These attractors exist only in dissipative systems.

Point attractor \(\rightarrow\) bounded curve
Sink or fixed point \(\rightarrow\) Limit cycle.
What types of attractors exist in non-linear systems?

1. In one-dimensional non-linear systems, in the absence of any external driving, sink and limit cycles are the only possible attractors.

2. More complicated attractors (called strange attractors — attractors with fractal properties) may exist provided,
   (a) 1-dimensional non-linear system is driven.
   (b) Non-linear system is higher dimensional.

(A) Therefore, nonlinearity is essential but not sufficient for complicated motion, called chaotic motion.

(B) Not every non-linear system will exhibit chaos.

(C) To see chaotic motion, the system (Hamiltonian or dissipative)
   (1) should be non-linear
   (2) should be described by at least 3, 1st order differential equations.
Project (at least 3-4 students)

1. Pick a system that we may exhibit chaos.
2. Has to be chosen before Thursday.
   (otherwise will not be allowed)
3. Get it approved by me.
4. Numerical solution
5. Study $\frac{\dot{x}}{x} = \frac{x}{t}$
   for different initial conditions
6. Phase portrait
7. Linear stability analysis
   a. For Hamilton, find Equil. pt
   b. For dissipative, find sink or
      saddle pt.
Gravitational:

\[ F = - \frac{G M m}{r^2} \quad \text{for a point mass} \]

\[ = - G m \int \frac{f(r)}{r^2} \, dv \]

Gravitational field:

\[ \vec{g} = \frac{F}{m} = - \frac{G M \hat{r}}{r^2} \]

\[ = - G \int \frac{f(r)}{r^2} \, dv \]

In 1-d: \[ F = - \frac{dV}{dx} \hat{x} \]

In general: \[ F = - \nabla \phi \]

Under what condition, \[ F = - \nabla \phi \]

\[ \Rightarrow \nabla \times \vec{F} = 0 \]

\[ \int \nabla \times \vec{F} = 0 \]