Dimer Decimation and Intricately Nested Localized-Ballistic Phases of a Kicked Harper Model

Tomaž Prosen, Indubala I. Satija, and Nausheen Shah

Physics Department, Faculty of Mathematics and Physics, University of Ljubljana, Jadranska 19, SI-1111 Ljubljana, Slovenia
Department of Physics, George Mason University, Fairfax, Virginia 22030

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A new decimation scheme is introduced to study localization transitions in tight binding models with long range interaction. Within this scheme, the lattice models are mapped to a vectorized dimer where an asymptotic dissociation of the dimer is shown to correspond to the vanishing of the transmission coefficient through the system. When applied to the kicked Harper model, the method unveils an intricately nested extended and localized phases in two-dimensional parameter space. In addition to computing transport characteristics with extremely high precision, the renormalization tools also provide a new method to compute quasienergy spectrum.

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The kicked Harper model has emerged as an important model in recent literature on quantum chaos [1–3]. The model exhibits both ballistic/extended as well as localized states in the regime where the corresponding classical system is chaotic. This challenges the concept of dynamical localization in nonintegrable systems (such as kicked rotor), interpreted as the suppression of quantum diffusion due to classical chaos. Localization-delocalization transition of the model although it has its roots in the corresponding integrable system, is very complex with mixed spectrum and nontrivial multifractal characteristics [2,3]. We use renormalization group (RG) techniques to demonstrate that the time-dependent problem exhibits intricately nested localized and ballistic states where the critical states exist only along the localization-delocalization boundary. Our RG equations are exact and facilitate studying systems of sizes up to $10^9$ with machine precision being the only bottleneck. The methodology has clear conceptual and numerical advantages over earlier methods for spotting critical states and locating localization transitions and can also be used to compute quasienergy spectrum.

The kicked Harper model [1] is described by the time-dependent Hamiltonian

$$H(t) = L \cos(p) + K \cos(q) \sum_{k=-\infty}^{\infty} \delta(t - k).$$

Here $q, p$ is a canonically conjugate pair of variables on a cylinder. In the absence of kicking, the quantum system $H_0 = L \cos(p) + K \cos(q)$, can be written as a nearest-neighbor (nn) lattice model. We take $p = m \hbar$ and $e^{i q}$ as a translational operator for $p$, obtaining the well-known Harper equation [4], $\frac{\sigma}{2} K (\psi_{m+1} + \psi_{m-1}) + L \cos(\hbar m) \psi_m = \epsilon \psi_m$. If we choose $\hbar = 2 \pi \sigma$, where $\sigma$ is an irrational with good Diophantine properties (i.e., badly approximated by rational numbers), the model exhibits localization-delocalization transition at $K = L$ [4]. This transition has been analyzed by various RG schemes [5–7]. In this Letter, we propose renormalization schemes for distinguishing extended, localized, and critical states in the time-dependent Harper model with almost the same precision as the corresponding time-independent problem.

For the time-periodic Hamiltonian (1) the matrix elements of the evolution operator $U$, in the angular-momentum basis $|m\rangle$ with eigenvalues $p = m \hbar$, read

$$U_{m,m'} = \exp[-2i L \cos(m \hbar)](-i)^{m-m'} J_{m-m'}(2 \tilde{K}).$$

Here $J_r$ is a Bessel function of order $r$ and $\tilde{K} = K/(2 \hbar)$ and $\tilde{L} = L/(2 \hbar)$. The spectral problem $U |\psi_\omega\rangle = e^{-i \omega t} |\psi_\omega\rangle$ (\omega being the quasienergy) involving unitary matrix of the above form was transformed [8] to a real lattice tight binding model (tbm) with angular momentum quantum number representing the lattice index

$$\sum_{r=-\infty}^{\infty} B_r^m u_{m+r} = 0,$$

where the coefficients $B_r^m$ are

$$B_r^m = J_r(\tilde{K}) \sin[\tilde{L} \cos(m \hbar) - \pi r/2 - \omega/2].$$

For finite $\tilde{K}$, the tbm (3) effectively contributes only a few terms as Bessel’s function exhibit fast decay when $|r| > |\tilde{K}|$. Therefore, the tbm describes a lattice model with a finite range of interaction denoted as $b$ ($b = \tilde{K}$). In the limit of small $\tilde{K}, \tilde{L}, \omega$, tbm reduces to the simple Harper equation with $\epsilon = \hbar \omega$ [4].

We introduce a new RG approach to study localization transitions in lattice models with long range interaction. The RG scheme although exact provides a machine precision tool to locate localization thresholds provided long range coupling decays asymptotically, and hence is ideal to study kicked Harper (1) represented by the lattice model (3),(4). We propose two independent RG schemes, which are a generalization of the dimer decimation scheme of the nn case [7] which reduces the lattice problem to a (vector, or block) dimer, the discrete analog of the textbook example of the quantum barrier problem. The transport characteristics of the lattice model can be understood as being due to quantum interference within the dimer. Furthermore, the decay of the coupling between the two sites...
of the dimer is shown to correspond to the vanishing of the transmission probability: i.e., the localized phase is asymptotically a broken dimer under the RG flow. As we discuss below, scaling analysis of the transmission properties provides an extremely accurate tool to distinguish extended, critical, and localized phases.

The key idea underlying both renormalization schemes is the simultaneous decimation of the two central sites of the doubly infinite lattice $-\infty, \ldots, -2, -1, 0, 1, 2, \ldots, \infty$, namely $\pm 1, \pm 2$, and so on, after we have eliminated the central site $m = 0$. Both can be applied to general tbm (3) with arbitrary coefficients $B^m_n$.

In the first scheme, which we will refer to as vector decimation, the decimation is done on a vectorized form of the tbm: the lattice model (3) is first transformed to a $nn$ vector model where each site is associated with a $b$-dimensional vector $\Phi_m$ [9]: $\Phi_m = (X^0_m, X^1_m, \ldots, X^{b-1}_m)$ where $X^0 = u_m$ and for $0 < r < b$, $X^r_m = B_m^{0-r} u_m + \sum_{k=0}^{r-1} B_m^{r-k} [u_m+r-k + u_m+r+k]$. The lattice model with long range coupling (3) can now be written as a $nn$ vector tbm, $\Phi_m + \Phi_{m+1} - V_m \Phi_m = 0$, where $V_m$ is a $b \times b$ matrix whose elements are related to $B^m_n$. We carry out the iterative process of decimating the two central sites of the vector tbm, after eliminating the central site of the lattice. At the $n$th step where all sites with $|m| < n$ have been eliminated, the tbm for $m = \pm n$ reads

$$\Phi_{n+1} + G(n)\Phi_{-n} - E(n)\Phi_n = 0, \quad (5)$$

$$\Phi_{-n-1} + G(n)\Phi_n - E(n)\Phi_{-n} = 0. \quad (6)$$

The renormalized matrices $G(n)$ and $E(n)$ are given by the following exact RG flow, the matrix version of the dimer map [7]

$$G(n + 1) = [E(n)G^{-1}(n)E(n) - G(n)]^{-1},$$

$$E(n + 1) = V_{n+1} + [G(n)E^{-1}(n)G(n) - E(n)]^{-1}. \quad (7)$$

In the second scheme, which we will refer to as scalar decimation, we seek a renormalization scheme which preserves the banded form of the tbm (3). Therefore, the main entity in this scheme is the $2b \times 2b$ central matrix $A$ after we decimate the central site, with the initial value $A_{j,k}(0) = B_{k-j}^j - B_{k-j}^0 / B_0^j$, $j, k \in \{-b, \ldots, -1, 1, \ldots, b\}$. The exact recursion describing the $n$th step renormalization of the matrix $A$ is

$$A_{j,k}(n + 1) = \det L_{j,k}(n)/D(n),$$

$$L_{j,k}(n) = \begin{pmatrix}
A_{j+k}^{j-k}(n) & A_{j-k}^{j+k}(n) \\
A_{j-k}^{-j+k}(n) & A_{j+k}^{-j-k}(n)
\end{pmatrix},$$

$D(n) = A_{-1,-1}(n)A_{1,1}(n) - A_{-1,1}(n)A_{1,-1}(n)$ is a central $2 \times 2$ determinant, $j^+ := j + 1, j - 1$ if $j > 0, < 0$, respectively, and

$$A_{j,k}^+(n) := \begin{cases}
A_{j,k}(n); & j, k \in \{-b, \ldots, -1, 1, \ldots, b\}, \\
B_{k-j}^{j^+}; & j > b \text{ or } k > b, \\
B_{k-j}^{-j^+}; & j < -b \text{ or } k < -b.
\end{cases}$$

We would like to emphasize that both the vector RG as well as the scalar RG flows are valid for the arbitrary value of $b$, the effective interaction range of the lattice representation of the tbm. However, numerical iteration of the flow requires choosing finite $b$ and results are found to be independent of $b$ provided $b = \bar{K}$. Comparing vector and scalar decimation flows, the latter has various numerical advantages in addition to being faster, as the only source of possible singularity, namely the central $2 \times 2$ determinant $D(n)$, is easy to control, and the whole procedure is completely stable against oversizing the bandwidth $b$.

However, two independent methods provide a unique advantage in confirming many subtle features of the phase diagram that are discussed below.

The important quantity that characterizes the transport and localization properties is the effective coupling of the renormalized dimer. It is the ratio of the off diagonal to the diagonal part of the renormalized tbm. To confirm that the renormalized coupling of the dimer is indeed related to the transmission coefficient of the model, we have done direct calculation of the transmission properties by solving the scattering problem on a momentum lattice for a truncated kicked model. This is achieved by replacing kinetic energy by $L \cos(kM)(M - |m|)$, $\theta(0) \geq 0 = : 1$, $\theta(0) = 0$, where the parameter $M$ is the size of the scattering region $|m| \leq M$. Outside the scattering region, $M = |M|$, the wave function is a superposition of properly normalized plane waves $\psi_m^{\pm}(l) = \sin(k_j)^{-1/2} \exp(\pm i k_j m)$, with $\cos(k_j) = (\omega + 2 \pi l)/ (2 \bar{K})$, integer. The reflection and transmission matrices $R$ and $T$ are determined by matching the Ansätze for the asymptotic solutions, $u_m = \psi_m^{\pm}(l) + \sum_n R_{ln} \psi_n^{\pm}(l)$, for $m < -M$, and $u_m = \sum_n T_{ln} \psi_n^{\pm}(l)$, for $m > -M$, on tbm (3) for $|m| \leq M$. Importantly, the decimation scheme makes the solution of the scattering problem for large $M$ very efficient, as $n = M - b$ iterates of the RG map (8) are performed first in order to maximally reduce the size of the scattering region, as $A(n)$ does not depend on truncation for $n \leq M - b$.

It turns out that the asymptotic value of the exponent $\beta(n) = \log P(n)/ \log n$, where $P(n) = \sum_{l, l'} |T_{ll'}|^2$ is the total transmission probability through the lattice of the (undecimated) size $M = n + b$, provides a very effective means to describe transport properties. The extended states are described by (typically monotonic) convergence of $\beta(n) \rightarrow 0$. In the case of exponential localization, $\beta(n) \rightarrow -\infty$. The decay $P(n) \sim \exp(-2n/\xi)$ can be used to calculate the localization length $\xi$. In contrast, the critical states are characterized by negative $\beta$ exhibiting nonconvergent, oscillatory behavior. In Fig. 1, we display RG flow at the Fibonacci iterates $F_j$ the successive denominators of the continued fraction.
FIG. 1. The RG flow at Fibonacci iterates, $F_f$. The monotonic curve with dots describes an extended state for $\bar{K} = 4$ and $L = 1$. The oscillatory pattern characterizing a critical state corresponds to Harper equation (diamond), kicked model with $\bar{K} = \bar{L} = 4$ (crosses). In addition, the critical state off the symmetry line is shown for $\bar{K} = 4.4$, $\bar{L} = 2.516711$ and its dual (short and long dashed lines with stars).

approximation of $\sigma = (\sqrt{5} - 1)/2$ (which is kept fixed throughout the paper). However, the method can be implemented for arbitrary $\sigma$. It should be noted that the RG results are independent of $b$ provided $b > \bar{K}$. All our results were obtained by choosing $b = [\bar{K}] + n_b$ with $n_b \sim 10$.

The main focus of this paper is to explore the variation in the transport properties in two-dimensional parameter space for a fixed value of the quasienergy, $\omega = 0$, which appears to be the eigenvalue for all parameter values. As we discuss later, the RG methodology can also be used to compute the quasienergy spectrum and analyze its transport characteristics.

The nonintegrability of kicked Harper is reflected throughout the two-dimensional parameter space. Along the symmetry line $K = L$, kicked Harper is found to remain critical for all values of $K$ and $\omega$. Here, the signature of nonintegrability is the variation in the exponent $\beta$ with $K$. However, for any value of $K$, the RG flow displays the oscillatory pattern of the type shown in Fig. 1 where the amplitude of the oscillation fluctuates strongly with $K$. For the $K \neq L$ case, we see a cascade of localization-delocalization transitions where the extended and the localized regimes display an intricate pattern in two-dimensional parameter space (Fig. 2). The extended regime of the Harper equation ($K > L$ regime) is landscaped by various patches of localized regimes, and similar behavior is seen in the localized phase consisting of patches of extended regime which is the dual image of the extended reentries. However, off the symmetry line, we do not see critical regions except along the localization-delocalization boundary. Based on various detailed numerics (where by iterating the RG equations up to $n = F_{35} = 14930352$, the localization thresholds are determined almost to machine precision), we conjecture that the two-dimensional phase diagram of kicked Harper exhibits localized and ballistics regions only where the critical manifold corresponding to singular continuous states is a multiply connected curve. We would like to point out that since some of the reentrant phases exist in an extremely narrow region of parameter space, RG iterations up to $F_{35}$ were necessary to establish that these regions did not correspond to critical phase as often seemed to be the case at lower RG iterates.

Figure 3 shows a one-dimensional projection of Fig. 2. We see a cascade of reentrant transitions where ballistic (localized) transport reappears and survives in a finite window in parameter after becoming localized (extended). We believe that these series of transitions are related to the cascades of transitions predicted by semiclassical methods [10]. We carried by a very extensive study of the localization-delocalization boundaries by various blow ups of the crossover regions. These crossover regions corresponding to extended-localized boundary showed a systematic shrinking as RG equations were iterated to higher and higher iterates (up to $F_{35}$).

We next show that the RG method can also be used to determine the spectral properties of the kicked model. Treating $\omega$ as a running parameter, RG flow together with
the duality transformation \((K, L) \rightarrow (L, K)\) (which maps extended states to localized states and vice versa as confirmed by detailed numerics) can also facilitate an accurate method to determine the quasienergy spectrum of the kicked Harper and related models (see Fig. 4). In the RG scheme, it is in general difficult to distinguish localized and forbidden values of \(\omega\), as it requires knowledge of \(d \beta / d \omega\). However, by exploiting duality, localized spectrum can be easily separated from the spectral gaps as the later correspond to having \(\beta(n) \rightarrow -\infty\) as well as \(\beta^{\text{dual}}(n) \rightarrow -\infty\).

Figure 4 shows the variation in the spectral characteristics as one of the parameter is varied. It is interesting that the RG method not only determines allowed quasienergies but also establishes its character (extended or localized) without relying on additional tests and conjectures [3,11]. The figure clearly establishes the mixed nature of the spectrum and together with Fig. 2 suggests an intricate nesting of extended and localized phases in the \(K - L - \omega\) space in kicked Harper. Following previous studies [11,12], we also investigated the question of finite measure of singular continuous states for fixed values of the parameters. Such states were ruled out by our analysis as higher iterates of RG flow exhibited oscillatory behavior only at the localization-delocalization boundary. In contrast to a qualitative approach [12], where various spectral blowups were used to rule out finite measure of self-similar scaling regions, the RG scheme provides a quantitative tool to single out critical states.

In summary, the RG approach proves to be essential in establishing nested localized-extended phases and ruling out critical phase (except along the localization boundary) in the nonintegrable Harper model. The proposed RG scheme for investigating localization transitions can be applied to a variety of problems which include unitary models without reflection symmetry as well as two-dimensional systems such as the two particle Harper model. We hope that these tools will provide a new direction in resolving various important issues underlying the frontiers of the localization phenomenon in complex systems.

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[9] Here we assumed, for simplicity, that the lattice model exhibits reflection symmetry \(B^\omega = B^{* \omega}\), as is the case for kicked Harper.