Harmonic Motion

Introduction
In this lab you will study simple harmonic motion qualitatively in the laboratory and use a program run in Excel to find the mathematical description of the motion you observe. You will also verify Hooke's law briefly in Part I. For the harmonic motion in Part II, you will record position data for a mass on a spring clamped to your lab table. By importing the data to Excel and fitting the appropriate equation to your data, you can study the quantitative nature of the observed simple harmonic motion in Part III.

Materials
Part I: Spring, clamp, small mass hanger, gram masses, meter stick, Excel
Part II: Spring, clamp, large mass hanger, motion sensor, Data Studio, Excel
Part III: Excel

Reference
Giancoli, Physics 6th Edition: Chapter 11, sections: 1,2,3,5

Theory
Part I: Consider a spring that is hanging down vertically from a support. When no mass hangs on it, it will remain at a length, L. This value is the "unstretched" or "rest" length for the spring. When a mass is added to the end of the spring it will stretch a distance $\Delta L$. The equilibrium position for the mass is then equal to $L + \Delta L$.

What happens if one pulls down or pushes up on the spring? The spring exerts a restoring force which is proportional to the distance it is stretched,

$$F = -kx$$  \hspace{1cm} (1)

where $x$ is the distance the spring is stretched or compressed and $k$ is the spring stiffness constant. The negative sign indicates that the force points opposite to the direction of the displacement of the mass. This is known as Hooke's Law.

Parts II and III: The restoring force causes the mass to oscillate up and down when it is displaced from equilibrium and released. The period, $T$, of the oscillation for simple harmonic motion depends on the mass, $m$, and the spring stiffness constant, $k$, of the spring.

$$T = 2\pi \sqrt{\frac{m}{k}}$$  \hspace{1cm} (2)
Since the frequency of the spring is \( f = \frac{1}{T} \), we can derive easily from Equation (2):

\[
f = \frac{1}{T} = \frac{1}{2\pi} \sqrt{\frac{k}{m}}
\]  

(3)

This frequency is called the natural frequency of the spring/mass system.

When the spring is oscillating, what will the graphs of position, velocity and acceleration versus time look like? Since the position \( x(t) \) goes above and below the equilibrium position we expect some repetitive motion such as a sine or cosine wave. The motion depends upon the position and the natural frequency as described by the following relationship:

\[
x(t) = A \cos(2\pi f t + \phi) + x_o
\]  

(4)

where

- \( x \) is the displacement, or the distance from the equilibrium position for the mass,
- \( A \) is the amplitude, or maximum displacement of the oscillation,
- \( f \) is the natural frequency of oscillation,
- \( \phi \) is a phase shift --this corrects for the fact that you probably did not start at the zero point of the cosine curve and the graph is shifted left or right along the \( x \)-axis. This is easiest to see when comparing the graph to a cosine curve that is NOT shifted. When you estimate the phase shift for your data, there will be two values that will yield a fit: one negative and one positive.
- \( x_o \) is vertical offset necessary because the motion sensor is located above the oscillating mass.
- \( t \) is time.

When the mass is at its highest point, the velocity is zero as it changes direction and begins to fall back down. When it reaches its lowest position, it again slows and changes direction in the oscillatory cycle. Therefore, the velocity graph should be 'out of phase' with the position graph. When the position vs. time graph is at a maximum or minimum, the velocity graph will be crossing zero, when the velocity is at its maximum, the position will be crossing zero, or we can say that they are 90 degrees out of phase. We can write the equation down for the velocity as:

\[
v = -A(2\pi f \sin(2\pi f t + \phi))
\]  

(5)

It should be noted that because the maximum value of sine is always 1.0 (no matter what the angle), the sine values always range from -1.0 to 1.0. The amplitude of the velocity curve is therefore equal to

\[
A_v = A(2\pi f).
\]  

(6)
What about the acceleration? When the displacement is at its maximum, the restoring force and therefore the acceleration will be maximum in the opposite direction. Therefore, it is 90 degrees out of phase with the velocity, and 180 degrees out of phase with the position graph. The relationship between the three curves is shown in Fig. 1 above. We can write the equation for the acceleration as:

\[ a = -A(2\pi f)^2 \cos(2\pi ft + \phi) \]  

(7)

Similar to the velocity curve, the maximum value of a cosine curve is always 1, therefore the amplitude of the acceleration curve is

\[ A_a = A(2\pi f)^2. \]  

(8)

You may find it helpful to refer to your book (Section 11-3, page 295) in order to understand what is happening between the force, acceleration, velocity and position during simple harmonic motion.

Procedure

**Part I: Hooke's Law**

1. Hang a small mass hanger from the spring and hold a meter stick next to it to measure the height. Adjust the position of the meter stick so that the mass hanger is at the zero position of the meter stick.
2. Add masses of 5, 10, 15, 20, 25 and 30 grams to the mass hanger and record the position (x) of the hanger after each mass is added. For example, in Fig. 2, the mass hanger has been displaced roughly 11.2 cm due to the added mass.

3. Calculate the downward force \( F = mg \) exerted by the additional weights.

4. Estimate the uncertainty for the position measurements.

5. Plot \( F \) vs. \( x \) in Excel and use horizontal error bars to show the uncertainty in position.

6. Plot a trendline for the \( F \) vs. \( x \) graph, and display the equation. Is the graph linear?

7. Use the linear regression tool (Tools>Data Analysis>Regression) to determine the slope and the uncertainty/standard deviation of the slope. This slope is the spring stiffness constant, \( k \).

**Part II**: Relationship between position, velocity and acceleration.

1. Hang the 50 gram mass hanger on the spring and place the motion sensor on the floor below it as shown in Fig. 3. Set the sample rate on the motion sensor to 50 Hz.
2. To determine the natural frequency of the system, pull down very straight on the mass hanger and set it into motion.

3. **Start** taking data and look at your position vs. time graph for your motion sensor. Make sure it is always seeing the mass hanger, if you have large spikes in your data, it is not reading correctly. Start again and adjust the position of the motion sensor.

4. Once you have everything aligned for data taking, set the mass hanger in motion again. Press **Start** and watch the position vs. time graph let it run through several cycles.

5. Use the **Smart Tool** to measure the amount of time it takes for the position graph to complete one complete cycle. This is the period, $T$, for the oscillation.

6. Determine the frequency from the period obtained above. Record this value and compare it with the predictions of Equation 3.

7. Bring the position, velocity and acceleration vs. time graphs into one graph window as shown in Fig. 1. Choose **Overlay** from the **Graph Settings** button, on the right most side of the **Graphs** button panel. This will lay the graphs on top of on another. Match up the time scales and shift the graphs as needed for vertical offsets.

8. Examine the graphs to see if the phase relationships outlined in the **Theory** section above are seen—your graph should look similar to Fig. 1. Print out the overlay graph.

9. Make marks on the graph and write comments about phase relationships at the maxima, minima and at the zero level for the position, velocity and acceleration curves. Make sure to comment on whether the phase relationships are as expected at the indicated points. Refer to your text if necessary.
10. Measure the amplitude of the position, which is $A$ in Equations (4), (5) and (6). For the position, the amplitude is simply equal to $A$. According to Equations (6) and (8), the velocity and acceleration curves have amplitudes equal $A (2\pi f)$ and $A^2 (2\pi f)^2$, respectively. Compare the amplitudes for velocity and acceleration that you measure from the graph with the predictions of Equations (6) and (8). Include these calculations in your sample calculations for your lab report.

**Part III: Fitting the harmonic motion equation.**

1. Select a few cycles of your position data, copy and paste it into the existing Excel spreadsheet, Harmonic Motion 244.xls. This spreadsheet is designed to aid you with curve fitting. You should paste your data so that it replaces the existing data. You will probably need to adjust the number of rows to match your data.

2. In Excel, you will have your experimental data as well as a theoretical calculation based on Eqn. (4) with parameters that can be adjusted. Your experimental data will be given by the blue curve. The pink curve is the theory which you will fit to the data by adjusting parameters in the spreadsheet.

3. Type the value for the frequency you calculated in the appropriate cell (you calculated this in **Part II**, step 6).

4. Look at the default values for the amplitude, $A$, and the phase shift, $\phi$, as well as the vertical offset, $x_o$. Replace these default values with what you estimate to be the correct values for your data. Remember that amplitude will change the height of the wave, the phase will shift the curve to the right or left, and the vertical offset will raise and lower the graph in relation to the $x$-axis.

5. After examining the graph of your data, readjust the theoretical values for amplitude, frequency, phase and vertical offset in the spreadsheet until the experimental and theoretical graphs are as close as possible.

6. Once the fit is as close as possible, graph the entire data set with theory on the same graph.

**Sample Calculations**

Include a sample calculation with uncertainty propagation of the expected frequency for the simple harmonic motion. Use equation (3) setting $m = (0.050 \pm 0.01)$ kg and using the $k$ value and its uncertainty from the linear regression. Also include calculations of the amplitudes of the velocity (Eqn. (6)) and acceleration (Eqn. (8)) curves and compare with experimental amplitudes in a table. It is not necessary to do an uncertainty calculation for the amplitude calculations.