Introduction to Tensors

Contravariant and covariant vectors

Rotation in 2-space: \[ x' = \cos \theta \, x + \sin \theta \, y \]
\[ y' = -\sin \theta \, x + \cos \theta \, y \]

To facilitate generalization, replace \((x, y)\) with \((x^1, x^2)\)

Prototype contravariant vector: \( d\mathbf{r} = (dx^1, dx^2) \)

\[
\begin{align*}
dx^1' &= \frac{\partial x^1'}{\partial x^1} \, dx^1 + \frac{\partial x^1'}{\partial x^2} \, dx^2 = \cos \theta \, dx^1 + \sin \theta \, dx^2 \\
\end{align*}
\]

Similarly for \( dx^2' \)

Same holds for \( \Delta \mathbf{r} \), since transformation is linear.
Compact notation: \[ dx^i' = \sum_j \frac{\partial x^i'}{\partial x^j} \, dx^j \]

(generalizes to any transformation in a space of any dimension)

Contravariant vector: \[ a^i' = \sum_j \frac{\partial x^i'}{\partial x^j} \, a^j \]

Now consider a scalar field \( \phi(\mathbf{r}) \): How does \( \nabla \phi \) transform under rotations?

\[
\nabla \phi = \left( \frac{\partial \phi}{\partial x^1}, \frac{\partial \phi}{\partial x^2} \right)
\]

\[
\frac{\partial \phi}{\partial x^i'} = \sum_j \frac{\partial \phi}{\partial x^j} \frac{\partial x^j}{\partial x^i'}
\]

\[
\nabla' \phi = \left( \frac{\partial \phi}{\partial x'^1}, \frac{\partial \phi}{\partial x'^2} \right)
\]

\[
\frac{\partial x^j}{\partial x'^i} \text{ appears rather than } \frac{\partial x^i'}{\partial x^j}
\]
For rotations in Euclidean n-space:

\[
\frac{\partial x^j}{\partial x^i} = \frac{\partial x^j'}{\partial x^i} = \cos \theta \quad \text{where } \theta = \text{angle btwn } x^j \text{ and } x^i' \text{ axes}
\]

It is not the case for all spaces and transformations that \( \frac{\partial x^j}{\partial x^i'} = \frac{\partial x^i'}{\partial x^j} \)

so we define a new type of vector that transforms like the gradient:

Covariant vectors: \( a^i' = \sum_j a_j \frac{\partial x^j}{\partial x^i} \)
Explicit demonstration for rotations in Euclidean 2-space:

\[ x^1' = \cos \theta \, x^1 + \sin \theta \, x^2 \]
\[ x^2' = -\sin \theta \, x^1 + \cos \theta \, x^2 \]
\[ x^1 = \cos \theta \, x^1' - \sin \theta \, x^2' \]
\[ x^2 = \sin \theta \, x^1' + \cos \theta \, x^2' \]

\[ \frac{\partial x^1'}{\partial x^1} = \cos \theta = \frac{\partial x^1}{\partial x^1'} \]
\[ \frac{\partial x^2'}{\partial x^1} = -\sin \theta = \frac{\partial x^1}{\partial x^2'} \]
\[ \frac{\partial x^1'}{\partial x^2} = \sin \theta = \frac{\partial x^2}{\partial x^1'} \]
\[ \frac{\partial x^2'}{\partial x^2} = \cos \theta = \frac{\partial x^2}{\partial x^2'} \]
What about vectors in Minkowski space?

\[ x^1' = \gamma x^1 - \gamma \beta x^4 \quad \quad x^1 = \gamma x^1' + \gamma \beta x^4' \]

\[ x^2' = x^2 \quad \quad x^2 = x^2' \]

\[ x^3' = x^3 \quad \quad x^3 = x^3' \]

\[ x^4' = -\gamma \beta x^1 + \gamma x^4 \quad \quad x^4 = \gamma \beta x^1' + \gamma x^4' \]

\[ \frac{\partial x^1'}{\partial x^4} = -\gamma \beta \quad \text{but} \quad \frac{\partial x^4}{\partial x^1'} = \gamma \beta \]

=> contravariant and covariant vectors are different!
Recap (for arbitrary space and transformation)

Contravariant vector: \[ A_i' = \sum_j \frac{\partial x_i'}{\partial x^j} A^j = \sum_j p^j_i A^j \]

Covariant vector: \[ A_i = \sum_j \frac{\partial x^j}{\partial x^i} A_j = \sum_j p^i_j A_j \]

For future convenience, define new notation for partial derivatives:

\[ p_i^i \equiv \frac{\partial x_i'}{\partial x^i} ; \quad p_i^j \equiv \frac{\partial x^i}{\partial x^i'} ; \quad \frac{\partial^2 x_i'}{\partial x^i \partial x^j} = p^i_{ij} \]

Note: \[ p_i^i = \sum_{i'} p_{i'}^i p_{i'}^i \quad ; \quad \sum_{i'} p_{i'}^i p_{i'}^j = \delta^i_j \]

\[ \delta^i_j = \text{Kronecker delta} = 1 \text{ if } i=j \quad , \quad 0 \text{ if } i \neq j \]
Tensors

Consider an $\mathcal{N}$-dimensional space (with arbitrary geometry) and an object with components $A^{i_1 \cdots i_k}_{l_1 \cdots l_n}$ in the $\{x^i\}$ coord system and $A^{i'_1 \cdots i'_k}_{l'_1 \cdots l'_n}$ in the $\{x^{i'}\}$ coord system.

This object is a mixed tensor, contravariant in $i\ldots k$ and covariant in $l\ldots n$, under the coord transformation $\{x^i\} \rightarrow \{x^{i'}\}$ if

$$A^{i'_1 \cdots i'_k}_{l'_1 \cdots l'_n} = \sum_{i_1 \cdots i_k, l_1 \cdots l_n} A^{i_1 \cdots i_k}_{l_1 \cdots l_n} p^{i'_1}_{i_1} \ldots p^{i'_k}_{i_k} p^{l'_1}_{l_1} \ldots p^{l'_n}_{l_n}$$

Rank of tensor, $M = \text{number of indices}$

Total number of components $= N^M$

Vectors are first rank tensors and scalars are zero rank tensors.
If space is Euclidean $N$-space and transformation is rotation of Cartesian coords, then tensor is called a “Cartesian tensor”.

In Minkowski space and under Poincaré transformations, tensors are “Lorentz tensors”, or, “4-tensors”.

Zero tensor $\mathbf{0}$ has all its components zero in all coord systems.

**Main theorem of tensor analysis:**

If two tensors of the same type have all their components equal in one coord system, then their components are equal in all coord systems.

**Einstein's summation convention:** repeated upper and lower indices $\Rightarrow$ summation

\[ A^i B^i = \sum_{i=1}^{N} A_i B^i \]
$A_i B^i$ could also be written $A_j B^j$; index is a “dummy index”

Another example: $A^{ij} B^k_j = \sum_{j=1}^N \sum_{k=1}^N A^{ij} B^k_j$

$j$ and $k$ are dummy indices; $i$ is a “free index”

Summation convention also employed with $\frac{\partial u^i}{\partial x^i}$, $\frac{\partial q}{\partial x^i} \frac{dx^i}{d\tau}$, etc.

Example of a second rank tensor: Kronecker delta

$$\delta^i_j p^{i'}_i p^{j'}_j = p^{i'}_j p^{j'}_j = \delta^{i'}_j$$
Tensor Algebra  (operations for making new tensors from old tensors)

1. Sum of two tensors: add components:  \( C_{k\ldots}^{i\ldots} = A_{k\ldots}^{i\ldots} + B_{k\ldots}^{i\ldots} \)

   Proof that sum is a tensor:  
   \[
   C_{k'}^{i'} = A_{k'}^{i'} + B_{k'}^{i'} = A_k^i p_{i}^{k'} + B_k^i p_{i}^{k'} 
   \]
   \[
   = (A_k^i + B_k^i) p_{i}^{k'} = C_k^i p_{i}^{k'} \]

2. Outer product: multiply components: e.g.,  \( C_{k\ldots lm}^{i\ldots j} = A_k^i B_{lm}^{j} \)

3. Contraction: replace one superscript and one subscript by a dummy index pair

   e.g.,  \( B_{km}^{ij} = A_{k hm}^{lij} \)

   Result is a scalar if no free indices remain.

   e.g.,  \( A_i^i \),  \( A_{ij}^{ij} \),  \( \delta_i^i = N \)
4. Inner product: contraction in conjunction with outer product

\[ C_{ikl} = A_{ij} B^{jl}_{kl} \]

Again, result is a scalar if no free indices remain, e.g., \( A_{ij} B^{ij} \)

5. Index permutation: e.g., \( B_{ijk} = A_{ikj} \)

Differentiation of Tensors

Notation: \( A^{i \ldots k}_{l \ldots n, r} \equiv \frac{\partial}{\partial x^r} \left( A^{i \ldots k}_{l \ldots n} \right) \); \( A^{i \ldots k}_{l \ldots n, rs} \equiv \frac{\partial^2}{\partial x^r \partial x^s} \left( A^{i \ldots k}_{l \ldots n} \right) \), etc.
\[ A_{l'...n',r'}^{i'...k'} = \frac{\partial}{\partial x^r} \left( A_{l...n}^{i...k} p_{i'}^{l'} ... p_{k'}^{l'} p_{l'}^{l} ... p_{n'}^{n} \right) \]

\[ = \frac{\partial}{\partial x^r} \left( A_{l...n}^{i...k} p_{i'}^{l'} ... p_{k'}^{l'} p_{l'}^{l} ... p_{n'}^{n} \right) p_{r'} \]

\[ = A_{l...n,r}^{i...k} p_{i'}^{l'} ... p_{k'}^{l'} p_{l'}^{l} ... p_{n'}^{n} p_{r'} \quad \text{IF transformation is linear} \]

(so that p's are all constant)

=> derivative of a tensor wrt a coordinate is a tensor only for linear transformations (like rotations and LTs)

Similarly, differentiation wrt a scalar (e.g., \( \tau \)) yields a tensor for linear transformations.
Now specialize to Riemannian spaces

characterized by a metric \( ds^2 = g_{ij} \, dx^i \, dx^j \) with \( \det(g_{ij}) \neq 0 \)

Assume \( g_{ij} \) is symmetric: \( g_{ij} = g_{ji} \) (no loss of generality, since they only appear in pairs)

If \( ds^2 > 0 \) when \( dx^i \neq 0 \), then space is “strictly Riemannian”
  (e.g., Euclidean \( N \)-space)

Otherwise, space is “pseudo-Riemannian” (e.g., Minkowski space)

\( g_{ij} \) is called the “metric tensor”.

Note that the metric tensor may be a function of position in the space.
Proof that $g_{ij}$ is a tensor:

$$g_{ij} dx^i dx^j = g_{ij} dx^{k'} p_{k'}^i dx^{l'} p_{l'}^j,$$

(since $dx^i$ is a vector)

$$ds^2 = g_{ij} dx^i dx^j = g_{k'l'} dx^{k'} dx^{l'},$$

(2 sets of dummy indices)

$$\Rightarrow (g_{k'l'} - g_{ij} p_{k'}^i p_{l'}^j) dx^{k'} dx^{l'} = 0$$

It's tempting to divide by $dx^{k'} dx^{l'}$ and conclude $g_{k'l'} = g_{ij} p_{k'}^i p_{l'}^j$.

But there's a double sum over $k'$ and $l'$, so this isn't possible.

Instead, suppose $dx^{i'} = 1$ if $i' = 1$

$$= 0 \text{ otherwise}$$

$$\Rightarrow g_{1'1'} - g_{ij} p_{1'}^i p_{1'}^j = 0 \quad \text{Similarly for} \quad g_{2'2'}, \text{ etc.}$$
\((g_{k'l'} - g_{ij} p^i_{k'} p^j_{l'}) dx^{k'} dx^{l'} = 0\)

Now suppose \(dx^{i'} = 1\) if \(i' = 1\) or \(2\)
\(= 0\) otherwise

Only contributing terms are:
- \(k'=1, l'=1\)
- \(k'=1, l'=2\)
- \(k'=2, l'=1\)
- \(k'=2, l'=2\)

\((g_{k'l'} - g_{ij} p^i_{k'} p^j_{l'}) dx^{k'} dx^{l'} = g_{1'1'} - g_{ij} p^i_{1'} p^j_{1'} + g_{2'2'} - g_{ij} p^i_{2'} p^j_{2'} +\)
\(g_{1'2'} - g_{ij} p^i_{1'} p^j_{2'} + g_{2'1'} - g_{ij} p^i_{2'} p^j_{1'}\)

\(g_{1'2'} = g_{2'1'}\) since \(g_{ij}\) is symmetric.

\(g_{ij} p^i_{2'} p^j_{1'} = g_{ij} p^i_{1'} p^j_{2'}\) since \(i\) and \(j\) are dummy indices.

\(\Rightarrow 2 \left( g_{1'2'} - g_{ij} p^i_{1'} p^j_{2'} \right) = 0\)
Similarly for all \(g_{i'j'}\) \((i' \neq j')\)
General definition of the scalar product: \( \mathbf{A} \cdot \mathbf{B} = g_{ij} A^i B^j \)

Define \( g^{ij} \) as the inverse matrix of \( g_{ij} : \ g^{ij} g_{jk} = \delta^i_k \)

\( g^{ij} \) is also a tensor, since applying tensor transformation yields

\[ g^{i'}{}^{j'} g_{j'k'} = \delta^{i'}_{k'} \], which defines \( g^{i'}{}^{j'} \) as the inverse of \( g_{i'j'} \)

Raising and lowering of indices: another tensor algebraic operation, defined for Riemannian spaces = inner product of a tensor with the metric tensor

e.g.: \( A_i = g_{ij} A^j \); \( A^i = g^{ij} A_j \) ; \( A^i_{\ jk} = g^{ir} g_{ks} A_r_{\ js} \)

Note: covariant and contravariant indices must be staggered when raising and lowering is anticipated.
4-tensors

In all coord systems in Minkowski space:

$$ds^2 = g_{\mu\nu} \, dx^\mu \, dx^\nu = c^2 dt^2 - dx^2 - dy^2 - dz^2$$

$$\Rightarrow \quad g_{\mu\nu} = \text{diag}(-1, -1, -1, 1) = g^{\mu\nu}$$

e.g: \[ A_i = g_{i\mu} A^\mu = -A^i \quad (i = 1, 2, 3) \]

\[ A_4 = g_{4\mu} A^\mu = A^4 \]

\[ U^\mu = \gamma(u) (u, c) \quad \Rightarrow \quad U_\mu = \gamma(u) (-u, c) \]
Under standard Lorentz transformations:

\[ p_1^{1'} = p_4^{4'} = \gamma, \quad p_4^{1'} = p_1^{4'} = -\gamma \beta, \quad p_2^{2'} = p_3^{3'} = 1 \]

\[ p_1^{1'} = p_4^{4'} = \gamma, \quad p_4^{1'} = p_1^{4'} = \gamma \beta, \quad p_2^{2'} = p_3^{3'} = 1 \]

All the other \( p \)'s are zero.

e.g.: \[ A^{1'2'} = A^{\mu \nu} p_{\mu}^{1'} p_{\nu}^{2'} = A^{\mu 2} p_{\mu}^{1'} = \gamma (A^{12} - \beta A^{42}) \]