Reading: Jackson 1.10, 2.1 through 2.10

We seek methods for solving Poisson's eqn with boundary conditions. The mathematical techniques that we will develop have much broader utility in physics.

Consider a grounded conducting plane at $z = 0$ and a point charge $q$ at $\vec{x}' = (x', y', z')$.

- Dirichlet boundary conditions: $\Phi(z = 0) = 0$ and $\Phi \to 0$ as $r \to \infty$ in upper half-space

What is $\Phi(\vec{x})$ for $z > 0$?

It's not just $\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} \frac{q}{|\vec{x} - \vec{x}'|}$ since charge is induced on the conductor.

Consider, for a moment, a completely different problem: no conducting plane and 2 point charges: $q$ at $\vec{x}' = (x', y', z')$ and $-q$ at $\vec{x}'' = (x', y', -z')$
For this,

$$\Phi(x, y, z) = \frac{q}{4\pi\varepsilon_0} \left[ \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} - \frac{1}{\sqrt{(x-x')^2 + (y-y')^2 + (z+z')^2}} \right]$$

For $z = 0$, $\Phi(x, y, z) = 0$. And, $\Phi \to 0$ as $r \to \infty$.

The 1$^{\text{st}}$ term satisfies Poisson's eqn in upper half-space: it's the potential of a point charge at $\vec{x}'$.

Explicitly: $\nabla^2 \Phi_1 = \frac{q}{4\pi\varepsilon_0} (-4\pi) \delta(\vec{x} - \vec{x}') = -\frac{q \delta(\vec{x} - \vec{x}')}{\varepsilon_0} = -\frac{\rho}{\varepsilon_0}$

The 2$^{\text{nd}}$ term satisfies Laplace's eqn in upper half-space:

$$\nabla^2 \Phi_2 = \frac{q \delta(\vec{x} - \vec{x}'')}{\varepsilon_0} = 0 \text{ in upper half-space (since } \vec{x}'' \text{ does not lie in upper half-space).}$$

Thus, $\nabla^2 \Phi = -\rho/\varepsilon_0$ in upper half-space and $\Phi$ satisfies the boundary conditions for the original problem, with the conducting plane at $z = 0$.

$\Rightarrow$ This $\Phi$ is the unique soln to the original problem.
This is called the “method of images”, since the “image charge” is placed at the location of the mirror image of \( q \) (for this simple geometry).

The attractive force on \( q \),
\[
F = -\frac{1}{4\pi\varepsilon_0} \frac{q^2}{(2z')^2}
\]

The work done in bringing charge \( q \) from \( \infty \) to \( z = z' \) is half the work done in bringing \( q \) and \( -q \) to distance \( 2z' \):

\[
W = -\frac{1}{4\pi\varepsilon_0} \frac{q^2}{4z'} \quad \text{(since } W = \frac{\varepsilon_0}{2} \int E^2 \, dV \text{ and } \vec{E} \text{ is only in half of space)}
\]

Explicit calculation:
\[
W = \int_{\infty}^{z'} \vec{F} \cdot d\vec{l} = \frac{q^2}{4\pi\varepsilon_0} \int_{\infty}^{z'} \frac{1}{z^2} \, dz = -\frac{1}{4\pi\varepsilon_0} \frac{q^2}{4z'}
\]

Suppose, instead of a single point charge, there were a charge distribution in the upper half-space. Then there is an image charge for each part of the distribution (works for both discrete and continuous dists).
Suppose the potential on the plane \( z = 0 \) is more complicated. For example, \( \Phi = V \) inside a circle of radius \( a \) centered on the origin and \( \Phi = 0 \) outside the circle. In this case, we can use a powerful method developed by Green.

Introduce Green functions \( G(\vec{x}, \vec{x}') \), which satisfy

\[
\nabla^2_{\vec{x}'} G(\vec{x}, \vec{x}') = -4\pi \delta(\vec{x} - \vec{x}')
\]

\[
\Rightarrow \quad G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} + F(\vec{x}, \vec{x}') \quad \text{where} \quad \nabla^2_{\vec{x}'} F(\vec{x}, \vec{x}') = 0
\]

Recall Green's Thm:

\[
\int_V (\phi \nabla^2_{\vec{x}'} \psi - \psi \nabla^2_{\vec{x}'} \phi) \, d^3 x' = \int_S \left[ \phi \frac{\partial \psi}{\partial n'} - \psi \frac{\partial \phi}{\partial n'} \right] \, da'
\]

With \( \phi = \Phi(\vec{x}') \) and \( \psi = G(\vec{x}, \vec{x}') \):

\[
\int_V \left[ \Phi(\vec{x}') (-4\pi) \delta(\vec{x} - \vec{x}') + G(\vec{x}, \vec{x}') \frac{\rho(\vec{x}')}{\epsilon_0} \right] \, d^3 x' = \int_S \left[ \Phi(\vec{x}') \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} - G(\vec{x}, \vec{x}') \frac{\partial \Phi(\vec{x}')}{\partial n'} \right] \, da'
\]

\[
\Rightarrow \quad \Phi(\vec{x}) = \frac{1}{4\pi \epsilon_0} \int_V \rho(\vec{x}') G(\vec{x}, \vec{x}') \, d^3 x' + \frac{1}{4\pi} \int_S \left[ G(\vec{x}, \vec{x}') \frac{\partial \Phi(\vec{x}')}{\partial n'} - \Phi(\vec{x}') \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} \right] \, da'
\]
For the Dirichlet problem, choose \( G(\vec{x}, \vec{x}') = G_D(\vec{x}, \vec{x}') \) such that \( G_D(\vec{x}, \vec{x}') = 0 \) for \( \vec{x}' \) on the bounding surface.

\[
\Rightarrow \quad \Phi(\vec{x}) = \frac{1}{4\pi \varepsilon_0} \int_V \rho(\vec{x}') G_D(\vec{x}, \vec{x}') \, d^3x' - \frac{1}{4\pi} \int_S \Phi(\vec{x}') \frac{\partial G_D(\vec{x}, \vec{x}')}{\partial n'} \, da'
\]

We know \( \rho(\vec{x}') \). If we can find \( G_D(\vec{x}, \vec{x}') \), then we can solve the problem for ANY Dirichlet boundary conditions [i.e., any \( \Phi(\vec{x}') \) on the surface]!

Returning to the image problem with the conducting plane: We have already found the Green function \( G(\vec{x}, \vec{x}') \)!

We found: With \( q \) at \( \vec{x}' \), potential at \( \vec{x} \) is given by

\[
\Phi(\vec{x}, \vec{x}') = \frac{q}{4\pi \varepsilon_0} \left[ \frac{1}{|\vec{x} - \vec{x}'|} - \frac{1}{|\vec{x} - \vec{x}'|} \right] \quad \text{with} \quad \vec{x}'' = (x', y', -z')
\]

\( \Phi(\vec{x}, \vec{x}') = 0 \) on the boundary and \( \nabla_x^2 \Phi = -\frac{q}{\varepsilon_0} \delta(\vec{x} - \vec{x}') \)

\[
\Rightarrow \quad G_D(\vec{x}, \vec{x}') = \frac{4\pi \varepsilon_0}{q} \Phi(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} - \frac{1}{|\vec{x} - \vec{x}''|}
\]
Thus, the physical meaning of the Dirichlet Green function is:

\[ 4\pi \varepsilon_0 / q \text{ times the potential at } \vec{x} \text{ due to (1) a point charge at } \vec{x}' \]

plus (2) whatever external charges are needed to make \( \Phi = 0 \) on the boundary (e.g.—charges induced on the surfaces, if they are conductors).

\[
\frac{\partial G_D}{\partial n'} = -\frac{\partial G_D}{\partial z'} \quad (\hat{n}' \text{ points AWAY from the region of interest})
\]

\[
= -\frac{\partial}{\partial z'} \left[ \frac{1}{|\vec{x} - \vec{x}'|} - \frac{1}{|\vec{x} - \vec{x}''|} \right]
\]

\[
\frac{\partial}{\partial z'} \left[ \frac{1}{|\vec{x} - \vec{x}''|} \right] = \frac{z - z'}{|\vec{x} - \vec{x}'|^3}
\]

\[
\frac{\partial}{\partial z'} \left[ \frac{1}{|\vec{x} - \vec{x}''|} \right] = \frac{\partial}{\partial z''} \left[ \frac{1}{|\vec{x} - \vec{x}''|} \right] \frac{\partial z''}{\partial z'} = -\frac{z - z''}{|\vec{x} - \vec{x}''|^3} = -\frac{z + z'}{|\vec{x} - \vec{x}''|^3}
\]

\[
\Rightarrow \quad \frac{\partial G_D}{\partial n'} = \frac{z' - z}{|\vec{x} - \vec{x}''|^3} - \frac{z' + z}{|\vec{x} - \vec{x}''|^3}
\]
On the boundary \( z' = 0 \):
\[
\frac{\partial G_D}{\partial n'} \bigg|_{z'=0} = \frac{-2z}{[(x-x')^2 + (y-y')^2 + z'^2]^{3/2}}
\]

If \( \Phi = V \) inside a circle of radius \( a \) (centered on the origin) and \( \Phi = 0 \) otherwise, then we should adopt cylindrical coords \((r, \phi, z)\).

\[
(x-x')^2 + (y-y')^2 = (r \cos \phi - r' \cos \phi')^2 + (r \sin \phi - r' \sin \phi')^2
= r^2 \cos^2 \phi + r'^2 \cos^2 \phi' - 2rr' \cos \phi \cos \phi' \\
+ r^2 \sin^2 \phi + r'^2 \sin^2 \phi' - 2rr' \sin \phi \sin \phi'
= r^2 + r'^2 - 2rr' \cos(\phi - \phi')
\]

\[
\frac{\partial G_D}{\partial n'} \bigg|_{z'=0} = \frac{-2z}{[z^2 + r^2 + r'^2 - 2rr' \cos(\phi - \phi')]^{3/2}}
\]

\[
-\frac{1}{4\pi} \oint_S \Phi(\vec{x}') \frac{\partial G_D}{\partial n'} \, d\alpha' = \frac{zV}{2\pi} \int_0^{2\pi} d\phi' \int_0^a r' \, dr' \left[ z^2 + r^2 + r'^2 - 2rr' \cos(\phi - \phi') \right]^{-3/2}
= \Phi(\vec{x}) \text{ when there is no charge in the upper half-plane. Otherwise, add }
\]

\[
\frac{1}{4\pi\varepsilon_0} \int_V \rho(\vec{x}') G_D(\vec{x}, \vec{x}') \, d^3x'
\]

to this.
Along the circle axis, $r = 0$:

$$\Phi = zV \int_0^a r' \, dr' \left[ z^2 + r'^2 \right]^{-3/2} = \frac{zV}{2} \int_{z^2}^{z^2+a^2} u^{-3/2} \, du = V \left[ 1 - \frac{z}{\sqrt{z^2 + a^2}} \right]$$

How to treat Neumann boundary conditions? Recall:

$$\Phi(\vec{x}) = \frac{1}{4\pi \varepsilon_0} \int_V \rho(\vec{x}') G(\vec{x}, \vec{x}') \, d^3 x' + \frac{1}{4\pi} \oint_S \left[ G(\vec{x}, \vec{x}') \frac{\partial \Phi(\vec{x}')}{\partial n'} - \Phi(\vec{x}') \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} \right] \, da'$$

It is tempting to choose $G(\vec{x}, \vec{x}')$ such that $\frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} = 0$ for $\vec{x}'$ on the surface.

But:

$$\int_V \nabla_x^2 G(\vec{x}, \vec{x}') \, d^3 x' = \int_V (-4\pi) \delta(\vec{x} - \vec{x}') \, d^3 x' = -4\pi$$

Also:

$$\int_V \nabla_x^2 G(\vec{x}, \vec{x}') \, d^3 x' = \int_V \nabla x' \cdot [\nabla x' G(\vec{x}, \vec{x}')] \, d^3 x' = \oint_S \hat{n} \cdot \nabla x' G(\vec{x}, \vec{x}') \, da' = \oint_S \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} \, da'$$

$$\Rightarrow \oint_S \frac{\partial G(\vec{x}, \vec{x}')}{\partial n'} \, da' = -4\pi \Rightarrow \frac{\partial G}{\partial n'} \equiv 0 \text{ on surface is impossible.}$$

(unless surface area $S$ is infinite)
Instead, take \( \frac{\partial G_N(\vec{x}, \vec{x}')}{\partial n'} = -\frac{4\pi}{S} \) (const) for \( \vec{x}' \) on surface, where \( S = \) total surf. area of boundary

\[
\Phi(\vec{x}) = \langle \Phi \rangle_S + \frac{1}{4\pi\varepsilon_0} \int_V \rho(\vec{x}') \, G_N(\vec{x}, \vec{x}') \, d^3 x' + \frac{1}{4\pi} \int_S \frac{\partial \Phi(\vec{x}')}{\partial n'} \, G_N(\vec{x}, \vec{x}') \, d\alpha'
\]

with \( \langle \Phi \rangle_S = \) the average of \( \Phi \) over the surface.

If the region of interest is infinite and \( \Phi \to 0 \) as \( r \to \infty \), then \( \langle \Phi \rangle_S = 0 \).

For the infinite plane, \( S = \infty \), so we want to enforce \( \partial G_N / \partial n = 0 \) everywhere.

To find \( G_N(\vec{x}, \vec{x}') \) for the infinite plane \( z' = 0 \), use the "anti-image":

\[
\begin{align*}
* q \\
\vdash \quad \quad E_z = 0 \\
* q \quad \text{(same sign)}
\end{align*}
\]

\[
E_z = -\frac{\partial \Phi}{\partial z} = \frac{\partial \Phi}{\partial n'} \Rightarrow \frac{\partial G_N}{\partial n'} = 0 \quad \text{for} \quad z' = 0
\]

\[
G_N(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} + \frac{1}{|\vec{x} - \vec{x}''|} = \frac{2}{\sqrt{z^2 + r^2 + r'^2 - 2rr' \cos(\phi - \phi')}}
\]

(the last equality holds when \( z' = 0 \))
Let's check that \( \int_S \frac{\partial G_N(\vec{x}, \vec{x}')}{\partial n'} da' = -4\pi \) for this example.

First, replace the upper half plane with a hemispherical volume, with radius \( R \), and place \( \vec{x} \) on the \( z \)-axis. We already know that \( \frac{\partial G_N}{\partial n'} = 0 \) for \( z' = 0 \), so the flat part of the hemisphere does not contribute to the surface integral.

To evaluate the derivative at \( r' = R \), express \( G_N \) in spherical coords.

\[
G_N = \left[ (x')^2 + (y')^2 + (z - z')^2 \right]^{-1/2} + \left[ (x')^2 + (y')^2 + (z + z')^2 \right]^{-1/2}
\]

\[
= \left[ (x')^2 + (y')^2 + (z')^2 + z^2 - 2zz' \right]^{-1/2} + \left[ (x')^2 + (y')^2 + (z')^2 + z^2 + 2zz' \right]^{-1/2}
\]

\[
= \left[ (r')^2 + z^2 - 2zr' \cos \theta' \right]^{-1/2} + \left[ (r')^2 + z^2 + 2zr' \cos \theta' \right]^{-1/2}
\]

\[
\frac{\partial G_N}{\partial n'} = \frac{\partial G_N}{\partial r'} = \frac{z \cos \theta' - r'}{\left[ (r')^2 + z^2 - 2zr' \cos \theta' \right]^{3/2}} - \frac{z \cos \theta' + r'}{\left[ (r')^2 + z^2 + 2zr' \cos \theta' \right]^{3/2}}
\]

\[
\int_S \frac{\partial G_N}{\partial n'} da' = 2\pi R^2 \int_0^{\pi/2} d\theta' \sin \theta' \left[ \frac{z \cos \theta' - R}{(R^2 + z^2 - 2zR \cos \theta')^{3/2}} - \frac{z \cos \theta' + R}{(R^2 + z^2 + 2zR \cos \theta')^{3/2}} \right] = -4\pi
\]

This holds for all \( R > z \) and the infinite plane corresponds to \( R \rightarrow \infty \).
To explicitly verify that $\langle \Phi \rangle_S = 0$, again consider a hemisphere with radius $R$.

On the plane $z' = 0$, \( \Phi(r') = \frac{q}{4\pi \varepsilon_0} \frac{2}{\sqrt{(r')^2 + z'^2}} \)

\[
\int da' \Phi(r') = \frac{q}{4\pi \varepsilon_0} \int_0^R \frac{4\pi r' dr'}{\sqrt{(r')^2 + z'^2}} = \frac{q}{\varepsilon_0} [\sqrt{R^2 + z'^2} - z] \to \frac{q}{\varepsilon_0} R \quad \text{as } R \to \infty.
\]

For the curved part of the hemisphere, \( \Phi(R) \approx \frac{q}{4\pi \varepsilon_0} \frac{2}{R} \quad \text{as } R \to \infty. \)

\[
\Rightarrow \quad \int da' \Phi(r') = \frac{q}{4\pi \varepsilon_0} \frac{2}{R} 2\pi R^2 = \frac{q}{\varepsilon_0} R
\]

Total surface area = \(3\pi R^2\)

\[
\Rightarrow \quad \langle \Phi \rangle_S \approx \frac{q}{\varepsilon_0} \frac{2R}{3\pi R^2} \to 0 \quad \text{as } R \to \infty.
\]
Suppose \( \partial \Phi / \partial n' = E_z \) within a circle of radius \( a \) centered on the origin and is zero outside. If there is no charge in the upper half-plane, then

\[
\Phi(\vec{x}) = \frac{E_z}{4\pi} \int_0^{2\pi} d\phi' \int_0^a r' \, dr' \frac{2}{\sqrt{z^2 + r^2 + r'^2} - 2 \, r \, r' \cos(\phi - \phi')}
\]

On the circle axis (\( r = 0 \)): \( \Phi(\vec{x}) = E_z \int_0^a \frac{r' \, dr'}{\sqrt{z^2 + r'^2}} = E_z \left[ \sqrt{z^2 + a^2} - z \right] \)

Check: \( \left. \frac{\partial \Phi}{\partial z} \right|_{z=0} = E_z \left[ \frac{z}{\sqrt{z^2 + a^2}} - 1 \right] \bigg|_{z=0} = -E_z \)

Symmetry of Green functions:

Green's Thm: \( \int_V (\phi \nabla_y^2 \psi - \psi \nabla_y^2 \phi) \, d^3y = \oint_S \left[ \phi \frac{\partial \psi}{\partial n} - \psi \frac{\partial \phi}{\partial n} \right] \, da \)

Take \( \phi = G(\vec{x}, \vec{y}) \) and \( \psi = G(\vec{x}', \vec{y}) \)

Recall \( \nabla_y^2 G(\vec{x}, \vec{y}) = -4\pi \delta(\vec{x} - \vec{y}) \)
For Dirichlet boundary conditions,

\[ \int_V \left[ G(\vec{x}', \vec{y}) (-4\pi) \delta(\vec{x}' - \vec{y}) - G(\vec{x}, \vec{y}) (-4\pi) \delta(\vec{x} - \vec{y}) \right] d^3y \]

\[ = \oint_S \left[ G(\vec{x}, \vec{y}) \frac{\partial G(\vec{x}', \vec{y})}{\partial n} - G(\vec{x}', \vec{y}) \frac{\partial G(\vec{x}, \vec{y})}{\partial n} \right] da \]

\[ \Rightarrow \quad G(\vec{x}, \vec{x}') - G(\vec{x}', \vec{x}) = \frac{1}{4\pi} \oint_S \left[ G(\vec{x}', \vec{y}) \frac{\partial G(\vec{x}, \vec{y})}{\partial n} - G(\vec{x}, \vec{y}) \frac{\partial G(\vec{x}', \vec{y})}{\partial n} \right] da \]

For Dirichlet boundary conditions,

\[ G_D(\vec{x}, \vec{y}) = 0 \quad \text{and} \quad G_D(\vec{x}', \vec{y}) = 0 \quad \text{for} \quad \vec{y} \quad \text{on the bounding surface} \]

\[ \Rightarrow \quad G_D(\vec{x}, \vec{x}') = G_D(\vec{x}', \vec{x}) \quad \text{"Green's Reciprocity Theorem"} \]

Recall: \( G_D(\vec{x}, \vec{x}') \propto \Phi \) at \( \vec{x} \) due to point charge at \( \vec{x}' \) plus external (e.g., induced) charge needed to satisfy boundary conditions. Reciprocity Thm

\[ \Rightarrow \quad \text{points of source (}\vec{x}') \text{ and observation (}\vec{x}) \text{ are interchangeable!} \]

Obviously true for an isolated charge with no boundaries except at \( \infty \). Remarkably, it remains true in the presence of conductors with fixed \( \Phi \).
For Neumann boundary conditions,

\[ G_N(\bar{x}, \bar{x}') - G_N(\bar{x}', \bar{x}) = \frac{1}{S} \oint_S [G_N(\bar{x}, \bar{y}) - G_N(\bar{x}', \bar{y})] \, da \]

\[ \Rightarrow \quad G_N(\bar{x}, \bar{x}') - \frac{1}{S} \oint_S G_N(\bar{x}, \bar{y}) \, da = G_N(\bar{x}', \bar{x}) - \frac{1}{S} \oint_S G_N(\bar{x}', \bar{y}) \, da \]

(symmetry in \( \bar{x} \) and \( \bar{x}' \))

Redefine \( G_N(\bar{x}, \bar{x}') \rightarrow G_N(\bar{x}, \bar{x}') - \frac{1}{S} \oint_S G_N(\bar{x}, \bar{y}) \, da \equiv f(\bar{x}, \bar{y}) \)

Since \( f(\bar{x}, \bar{y}) \) is independent of \( \bar{x}' \):

\[ \nabla_{\bar{x}'}^2 f(\bar{x}, \bar{y}) = 0 \quad \text{and} \quad \frac{\partial f(\bar{x}, \bar{y})}{\partial n'} = 0 \]

So, if the original \( G_N \) satisfied \( \nabla_{\bar{x}'}^2 G_N(\bar{x}, \bar{x}') = -4\pi \delta(\bar{x} - \bar{x}') \) and \( \frac{\partial G_N(\bar{x}, \bar{x}')}{\partial n'} = -\frac{4\pi}{S} \), then the new \( G_N \) will, too. Thus, we can choose \( G_N \) such that \( G_N(\bar{x}, \bar{x}') = G_N(\bar{x}', \bar{x}) \).
Note: The freedom to add a constant to \( G_N \) arises because only \( \partial G_N / \partial n' \) is specified on the boundary.

Another classic image problem: a charge \( q \) outside a grounded, conducting sphere of radius \( a \)

Suppose \( q \) is located at \( d \hat{x} \). Can we find an image charge inside the sphere such that \( \Phi = 0 \) on the sphere's surface \((|r| = a)\)? By symmetry, it must lie on the \( x \)-axis. So, take an image charge \( \alpha q \) located at \( b \hat{x} \).

\[
\Phi(\vec{r} = a \hat{r}) = \frac{q}{4\pi\varepsilon_0} \left[ \frac{1}{|a \hat{r} - d \hat{x}|} + \frac{\alpha}{|a \hat{r} - b \hat{x}|} \right]
\]

With \( \cos \gamma \equiv \hat{r} \cdot \hat{x} \),

\[
\frac{4\pi\varepsilon_0}{q} \Phi = \frac{1}{\sqrt{d^2 + a^2 - 2ad \cos \gamma}} + \frac{\alpha}{\sqrt{a^2 + b^2 - 2ba \cos \gamma}} = 0
\]

\[
\Rightarrow \quad \alpha^2 (d^2 + a^2 - 2ad \cos \gamma) = a^2 + b^2 - 2ba \cos \gamma
\]

\[
\alpha^2 (d^2 + a^2) - 2\alpha^2 ad \cos \gamma = a^2 + b^2 - 2ba \cos \gamma
\]
Must be true for all $\gamma \Rightarrow \alpha^2 (d^2 + a^2) = a^2 + b^2$ and $\alpha^2 d = b$

$\Rightarrow \alpha^2 (d^2 + a^2) = a^2 + \alpha^4 d^2$

$\Rightarrow \alpha^2 = \frac{(d^2 + a^2) \pm \sqrt{(d^2 + a^2)^2 - 4d^2a^2}}{2d^2} = \frac{(d^2 + a^2) \pm (d^2 - a^2)}{2d^2}$

Image charge must be inside sphere $\Rightarrow b < d \Rightarrow \alpha^2 < 1 \Rightarrow$ must choose neg sign

$\Rightarrow \alpha = -\frac{a}{d}$ (image charge must have opp sign as $q$ for $\Phi = 0$ on surface)

$b = \frac{a^2}{d}$ (Note: $\alpha$ and $b$ can easily be recalled with dimensional reasoning)

Caution: I'm following Jackson's notation, where $x'$ is mag of vector, not $x'$-component!

Dirichlet Green fcn: $G(x', \vec{x}^\prime) = \frac{1}{|\vec{x} - \vec{x}'|} + \frac{\alpha}{|\vec{x} - \vec{x}''|}$

with $\alpha = -\frac{a}{d} = -\frac{a}{|\vec{x}'|} = -\frac{a}{x'}$ ; $\vec{x}'' = \frac{a^2}{d} \hat{x}' = \frac{a^2}{x'^2} \vec{x}'$ ; $x'' = |\vec{x}''| = \frac{a^2}{x'}$
In spherical coords (with $\gamma$ the polar angle):

**Law of Cosines:**

$$|\vec{x} - \vec{x}'|^2 = x^2 + x'^2 - 2 xx' \cos \gamma$$

$$|\vec{x} - \vec{x}''|^2 = x^2 + x''^2 - 2 xx'' \cos \gamma$$

$$= x^2 + \frac{a^4}{x'^2} - 2 a^2 \frac{x}{x'} \cos \gamma$$

$$G(\vec{x}, \vec{x}') = \frac{1}{\sqrt{x^2 + x'^2 - 2 xx' \cos \gamma}} - \frac{a}{x'} \frac{1}{\sqrt{x^2 + \frac{a^4}{x'^2} - 2 a^2 \frac{x}{x'} \cos \gamma}}$$

$$= \frac{1}{\sqrt{x^2 + x'^2 - 2 xx' \cos \gamma}} - \frac{1}{\sqrt{\left(\frac{x}{a}\right)^2 + a^2 - 2 xx' \cos \gamma}}$$
\[
\left. \frac{\partial G}{\partial n'} \right|_{x' = a} = - \left. \frac{\partial G}{\partial x'} \right|_{x' = a} \\
= \left. - \frac{1}{2} \left( x^2 + x'^2 - 2 xx' \cos \gamma \right)^{-3/2} \left( 2x' - 2x \cos \gamma \right) 
+ \frac{1}{2} \left[ \left( \frac{x x'}{a} \right)^2 + a^2 - 2xx' \cos \gamma \right]^{-3/2} \left( 2 \frac{x^2}{a^2} x' - 2x \cos \gamma \right) \right|_{x' = a} \\
= \left( x^2 + a^2 - 2ax \cos \gamma \right)^{-3/2} (a - x \cos \gamma) - \left( x^2 + a^2 - 2ax \cos \gamma \right)^{-3/2} \left( \frac{x^2}{a} - x \cos \gamma \right) \\
= \frac{a - x^2/a}{(x^2 + a^2 - 2ax \cos \gamma)^{3/2}} = -\frac{x^2 - a^2}{a (x^2 + a^2 - 2ax \cos \gamma)^{3/2}}
\]

(normal points away from region of interest, i.e., from exterior to interior of sphere)
Soln of Laplace eqn is

\[
\Phi(\vec{x}) = -\frac{1}{4\pi} \int_S \Phi(\vec{x}') \frac{\partial G_D}{\partial n'} \, da' = -\frac{1}{4\pi} \int d\Omega' a^2 \Phi(a, \theta', \phi') \frac{\partial G_D}{\partial n'} \\
= \frac{1}{4\pi} \int d\Omega' \Phi(a, \theta', \phi') \frac{a(x^2 - a^2)}{(x^2 + a^2 - 2ax \cos \gamma)^{3/2}}
\]

\[
\gamma = \text{angle btwn } \vec{x} \text{ and } \vec{x}' \\
\vec{x} : (x, \theta, \phi) \quad \vec{x}' : (x', \theta', \phi')
\]

\[
\hat{x} \cdot \hat{x}' = (\hat{x})_x (\hat{x}')_x + (\hat{x})_y (\hat{x}')_y + (\hat{x})_z (\hat{x}')_z \\
= \sin \theta \sin \theta' \cos \phi \cos \phi' + \sin \theta \sin \theta' \sin \phi \sin \phi' + \cos \theta \cos \theta'
\]

\[
= \sin \theta \sin \theta' \cos(\phi - \phi') + \cos \theta \cos \theta'
\]

\[
\Rightarrow \cos \gamma = \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')
\]

Because of the complicated dependence of $\cos \gamma$ on $\theta$, $\theta'$, $\phi$, $\phi'$, the general case does not yield analytic results.
On the $z$-axis: \( \theta = 0 \Rightarrow \cos \gamma = \cos \theta' \Rightarrow x = z \)

\[
\Phi(z) = \frac{1}{4\pi} \int_0^\pi \sin \theta' \, d\theta' \int_0^{2\pi} d\phi' \Phi(a, \theta', \phi') \frac{a(z^2 - a^2)}{(z^2 + a^2 - 2az \cos \theta')^{3/2}}
\]

\[
= \frac{a(z^2 - a^2)}{2} \int_0^\pi \sin \theta' \, d\theta' \langle \Phi \rangle_{\phi'}(a, \theta')(z^2 + a^2 - 2az \cos \theta')^{-3/2}
\]

\(\langle \Phi \rangle_{\phi'}(a, \theta')\) is the average of \(\Phi\) over \(\phi'\).

Suppose \(\Phi = V\) (const) on the sphere.

\[
\Phi(z) = \frac{a(z^2 - a^2)V}{2(2az)} \int_{(z-a)^2}^{(z+a)^2} du \, u^{-3/2} = \frac{a(z^2 - a^2)V}{2az} \left[ \frac{1}{z-a} - \frac{1}{z+a} \right]
\]

\[
= \frac{a(z^2 - a^2)V}{2az} \frac{2a}{z^2 - a^2} = V \frac{a}{z}, \text{ which checks.}
\]

Symmetry \(\Rightarrow\) $z$-axis is equivalent to any other axis \(\Rightarrow\)

\(\Phi(r, \theta, \phi) = V \frac{a}{r}\)
Suppose $\Phi = +V$ on “Northern” hemisphere $(0 < \theta < \pi/2)$
and $\Phi = -V$ on “Southern” hemisphere $(\pi/2 < \theta < \pi)$.

Splitting integral into 2 pieces:

$$\Phi(z) = \frac{a(z^2 - a^2)V}{2az} \left[ \frac{1}{z-a} - \frac{1}{\sqrt{z^2 + a^2}} - \frac{1}{\sqrt{z^2 + a^2}} + \frac{1}{z+a} \right]$$

$$= \frac{(z^2 - a^2)V}{2z} \left[ \frac{2z}{z^2 - a^2} - \frac{2}{\sqrt{z^2 + a^2}} \right]$$

$$= V \left[ 1 - \frac{z^2 - a^2}{z \sqrt{z^2 + a^2}} \right]$$

(check: $= V$ when $z = a$)

Note: Given the solns for $\Phi = \text{const}$ and $\Phi = \pm V$ in the Northern/Southern hemispheres, superposition yields soln for any 2 different potentials in the 2 hemispheres.
This mathematical interlude is preparation for the next method of solving electrostatic boundary value problems—separation of variables.

Consider a set of functions \( U_n(\xi) \) \((n = 1, 2, 3, \ldots)\)

They are **orthogonal** on interval \( \xi \in (a, b) \) if

\[
\int_a^b U_n^*(\xi)U_m(\xi) \, d\xi = \begin{cases} 
0 & , m \neq n \\
\neq 0 & , m = n
\end{cases}
\]

* denotes complex conjugation: \((a + b i)^* = a - b i\)

Normalize so that the integral with \( n = m \) is unity. Then the functions are orthonormal:

\[
\int_a^b U_n^*(\xi)U_m(\xi) \, d\xi = \delta_{nm}
\]

The orthonormal functions \( U_n(\xi) \) are “**complete**” if any well-behaved fcn \( f(\xi) \) can be expressed with arbitrarily small error as a series of them.
In this case, \( f(\xi) = \sum_{n=1}^{\infty} a_n U_n(\xi) \)

The coefficients \( a_n \) can be found using this trick:

\[
\int_{a}^{b} U_n^*(\xi) f(\xi) \, d\xi = \int_{a}^{b} U_n^*(\xi) \sum_{m=1}^{\infty} a_m U_m(\xi) \, d\xi \\
= \sum_{m=1}^{\infty} a_m \int_{a}^{b} U_n^*(\xi) U_m(\xi) \, d\xi = \sum_{m=1}^{\infty} a_m \delta_{nm} = a_n
\]

So, \( a_n = \int_{a}^{b} U_n^*(\xi) f(\xi) \, d\xi \)

Thus, \( f(\xi) = \sum_{n=1}^{\infty} \int_{a}^{b} U_n^*(\xi') f(\xi') \, d\xi' U_n(\xi) = \int_{a}^{b} \left[ \sum_{n=1}^{\infty} U_n^*(\xi') U_n(\xi) \right] f(\xi') \, d\xi' \)

\[
\Rightarrow \sum_{n=1}^{\infty} U_n^*(\xi') U_n(\xi) = \delta(\xi' - \xi) \quad \text{("completeness relation")}
\]
Straightforward generalization to multiple dimensions. For example, for variables $\xi \in (a, b)$ and $\eta \in (c, d)$,

$$f(\xi, \eta) = \sum_{n,m} a_{nm} U_n(\xi) V_m(\eta) \quad \text{with} \quad a_{nm} = \int_{a}^{b} d\xi \int_{c}^{d} d\eta \ U_n^*(\xi) V_m^*(\eta) \ f(\xi, \eta)$$

Sines and cosines form a complete set of orthogonal functions. When a fcn is expressed in terms of them, it's called a “Fourier series”.

Useful trig identities:

$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$

$\cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)]$

$\sin A \cos B = \frac{1}{2} [\sin(A - B) + \sin(A + B)]$

Consider the interval $\left(-\frac{a}{2}, \frac{a}{2}\right)$

If $n \neq m$, 

\[
\int_{-a/2}^{a/2} \sin \left(\frac{2\pi n x}{a}\right) \sin \left(\frac{2\pi m x}{a}\right) \, dx
\]

\[
= \frac{1}{2} \int_{-a/2}^{a/2} \cos \left[\frac{2\pi (n - m) x}{a}\right] \, dx - \frac{1}{2} \int_{-a/2}^{a/2} \cos \left[\frac{2\pi (n + m) x}{a}\right] \, dx
\]

\[
= \frac{a}{4\pi(n - m)} \int_{-(n-m)\pi}^{(n-m)\pi} \cos u \, du - \frac{a}{4\pi(n + m)} \int_{-(n+m)\pi}^{(n+m)\pi} \cos u \, du = 0 - 0 = 0
\]

Similarly for integrals of \(\cos \left(\frac{2\pi n x}{a}\right) \cos \left(\frac{2\pi m x}{a}\right)\) and \(\sin \left(\frac{2\pi n x}{a}\right) \cos \left(\frac{2\pi m x}{a}\right)\)

**Normalization:**

\[
A^2 \int_{-a/2}^{a/2} \sin^2 \left(\frac{2\pi n x}{a}\right) \, dx = A^2 \frac{a}{2\pi n} \int_{-\pi n}^{\pi n} \sin^2 u \, du = \frac{A^2 a}{2\pi n} \left[\frac{1}{2} u - \frac{1}{4} \sin 2u\right]_{-\pi n}^{\pi n}
\]

\[
= \frac{A^2 a}{2} \quad \Rightarrow \quad \text{normalization constant} \quad A = \sqrt{\frac{2}{a}}
\]
Thus, the Fourier series expansion for a fcn $f(x)$ is:

$$f(x) = \frac{1}{2}A_0 + \sum_{m=1}^{\infty} \left[ A_m \cos \left( \frac{2\pi mx}{a} \right) + B_m \sin \left( \frac{2\pi mx}{a} \right) \right]$$

with

$$A_m = \frac{2}{a} \int_{-a/2}^{a/2} f(x) \cos \left( \frac{2\pi mx}{a} \right) \, dx \quad B_m = \frac{2}{a} \int_{-a/2}^{a/2} f(x) \sin \left( \frac{2\pi mx}{a} \right) \, dx$$

The $\frac{1}{2}$ is included in front of $A_0$ so that the above formula for $A_m$ will also work with $m = 0$. The interval $(-a/2, a/2)$ may be shifted by any constant $a_0$: $(a_0 - a/2, a_0 + a/2)$. e.g.: $(0, a)$.

**Note:** The constant term is needed for completeness. The proof of completeness can be extremely difficult, depending on how well-behaved we require the fcn $f(x)$ to be; we will omit it. (This is true of all the complete sets of orthogonal fcns that we will consider.)
The following alternative Fourier series are also complete:

1) \[ f(x) = \sum_{n=1}^{\infty} a_n \sin \left( \frac{\pi n x}{a} \right) \quad \text{with} \quad a_n = \frac{2}{a} \int_0^a f(x) \sin \left( \frac{\pi n x}{a} \right) \, dx \]
on interval \((0, a)\). \quad \text{NB: Not orthogonal on \((-a/2, a/2)\)!}

“Fourier sine series”

2) \[ f(x) = \frac{1}{2} a_0 + \sum_{n=1}^{\infty} a_n \cos \left( \frac{\pi n x}{a} \right) \quad \text{with} \quad a_n = \frac{2}{a} \int_0^a f(x) \cos \left( \frac{\pi n x}{a} \right) \, dx \]
on interval \((0, a)\). \quad \text{NB: Not orthogonal on \((-a/2, a/2)\)!}

3) \[ f(x) = \sum_{n=-\infty}^{\infty} a_n e^{i 2\pi n x/a} \quad \text{with} \quad a_n = \frac{1}{a} \int_{-a/2}^{a/2} f(x) e^{-i 2\pi n x/a} \, dx \]

Example: \[ f(x) = x \quad \text{on} \quad -\frac{a}{2} < x < \frac{a}{2} \quad \text{A}_0 = \frac{2}{a} \int_{-a/2}^{a/2} x \, dx = 0 \]
\[ A_m = \frac{2}{a} \int_{-a/2}^{a/2} x \cos \left( \frac{2\pi mx}{a} \right) \, dx = \frac{a}{2\pi^2 m^2} \int_{-m\pi}^{m\pi} u \cos u \, du \]

\[ = \frac{a}{2\pi^2 m^2} \left[ x \sin x + \cos x \right]_{-m\pi}^{m\pi} = 0 \]

=> none of the cosines contribute. As expected, since \( \cos(2\pi mx/a) \) are even functions and \( f(x) = x \) is odd.

\[ B_m = \frac{2}{a} \int_{-a/2}^{a/2} x \sin \left( \frac{2\pi mx}{a} \right) \, dx = \frac{a}{2\pi^2 m^2} \left[ \sin x - x \cos x \right]_{-m\pi}^{m\pi} = \frac{a}{\pi m} (-1)^{m+1} \]

So, \( x = \frac{a}{\pi} \sum_{m=1}^{\infty} (-1)^{m+1} \frac{1}{m} \sin \left( \frac{2\pi mx}{a} \right) \) for \( -\frac{a}{2} < x < \frac{a}{2} \)
10 terms
30 terms
Separation of Variables

Some solns of the Laplace eqn are of the form \( \Phi(x, y, z) = X(x) Y(y) Z(z) \)

For example: \( \Phi = A \, x \, y \, z \)

[But not all solns are of this form; e.g., \( \Phi = A (x + y + z) \)]

\[
\frac{1}{\Phi} \nabla^2 \Phi = \frac{1}{X Y Z} \left( Y Z \frac{\partial^2 X}{\partial x^2} + X Z \frac{\partial^2 Y}{\partial y^2} + X Y \frac{\partial^2 Z}{\partial z^2} \right) \\
= \frac{1}{X} \frac{\partial^2 X}{\partial x^2} + \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} + \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = 0
\]

Each of the 3 terms is a fcn of just 1 variable—\( x, y, \) or \( z \).

\( \Rightarrow \) Each term must be a constant.

Otherwise, if the sum = 0 for one value of \( x \), then it won't for a different value of \( x \), since only the first term would have changed. But Laplace eqn must hold for all \( x \). Likewise for \( y \) and \( z \).
So: \[ \frac{1}{X} \frac{\partial^2 X}{\partial x^2} = C_1 \quad \frac{1}{Y} \frac{\partial^2 Y}{\partial y^2} = C_2 \quad \frac{1}{Z} \frac{\partial^2 Z}{\partial z^2} = C_3 \]

\[ C_1 + C_2 + C_3 = 0 \]

The problem got vastly easier, since a PDE was replaced with 3 ODEs! The solutions are sines, cosines, and exponentials. When the boundary surfaces are planes, a sum of “separated” solns may satisfy the boundary conditions.

There are 11 known coord systems for which the Laplace eqn is separable (i.e., for which \( \Phi \) can be expressed as a product of 3 functions of a single coordinate with the Laplace eqn reducing to an ODE for each function). We'll discuss spherical and cylindrical coords later.

**Example in Cartesian coords:** Rectangular box with lengths \( a, b, c \) in the \( x, y, z \) directions and one corner at the origin.

\[ \Phi = 0 \text{ on all the faces except } z = c, \]
where \( \Phi = V(x,y) \).

Find \( \Phi \) everywhere inside the box.
Since $\Phi = 0$ at both ends in $x$ and $y$, $X$ and $Y$ need to be sines; no difference of exponentials can vanish for 2 different values of $x$.

$\Rightarrow$ $C_1$ and $C_2$ are negative

Take $C_1 = -\alpha^2$ and $C_2 = -\beta^2$ $\Rightarrow$ $C_3 = \alpha^2 + \beta^2$

Soln of $\frac{\partial^2 X}{\partial x^2} = -\alpha^2 X$ is $X \propto \sin(\alpha x + \theta_x)$

$X = 0$ at $x = 0$ $\Rightarrow$ $\theta_x = 0$

$X = 0$ at $x = \alpha$ $\Rightarrow$ $\sin \alpha a = 0$ $\Rightarrow$ $\alpha a = 0, n\pi, 2n\pi, \ldots$ $\Rightarrow$ $\alpha_n = \frac{n\pi}{a}$

Similarly for $Y$: $Y \propto \sin(\beta_m y)$ with $\beta_m = \frac{m\pi}{b}$

$\frac{\partial^2 Z}{\partial z^2} = (\alpha_n^2 + \beta_m^2)Z$ Solns are $Z \propto e^{\pm \sqrt{\alpha_n^2 + \beta_m^2} z}$

$Z = 0$ at $z = 0$ $\Rightarrow$ $Z \propto e^{\sqrt{\alpha_n^2 + \beta_m^2} z} - e^{-\sqrt{\alpha_n^2 + \beta_m^2} z} \propto \sinh \left( \sqrt{\alpha_n^2 + \beta_m^2} z \right)$
Thus, separated solns that satisfy 5 of the 6 boundary conditions have the form

\[ A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} z) \]

with \( \alpha_n = \frac{n\pi}{a} \), \( \beta_m = \frac{m\pi}{b} \), \( \gamma_{nm} = \sqrt{\alpha_n^2 + \beta_m^2} = \pi \sqrt{\left(\frac{n}{a}\right)^2 + \left(\frac{m}{b}\right)^2} \)

Since the Laplace eqn is linear, linear combinations of solns are also solns. Can we find a linear combination of solns of the above separated form that will satisfy the 6th boundary condition? We require

\[ V(x, y) = \sum_{n,m=1}^{\infty} A_{nm} \sin(\alpha_n x) \sin(\beta_m y) \sinh(\gamma_{nm} c) \]

This is a 2D Fourier sine series. Any \( V(x,y) \) can be expanded this way.

\[ A_{nm} = \frac{4}{ab \sinh(\gamma_{nm} c)} \int_0^a dx \int_0^b dy V(x, y) \sin(\alpha_n x) \sin(\beta_m y) \]