Class #17

Last Class:
Vector Potential $A$

$$
\nabla^2 A = -4\pi J \quad \text{(instead of Ampère's law $\nabla \times B = \mu_0 J$)}
$$

Multipole Expansion of $A$

$$
\vec{A}(r) = \frac{\mu_0 I}{4\pi} \int \frac{1}{r'} d\vec{l}'' = \frac{\mu_0 I}{4\pi} \sum_{n=0}^{\infty} \frac{1}{r^{n+1}} \int (r')^n P_n (\cos \theta') d\theta''
$$

or

$$
\vec{A}(r) = \frac{\mu_0 I}{4\pi} \left( \frac{1}{r} \int d\vec{l}'' + \frac{1}{r^2} \int r' \cos \theta' d\theta'' + \frac{1}{r^3} \int (r')^2 \left( \frac{3 \cos^2 \theta' - 1}{2} \right) d\theta'' \right)
$$

$$
= \text{monopole} + \text{dipole} + \text{quadrupole etc...}
$$

TODAY:
Magnetic Fields in Matter (6.1)

Magnetization:

- Some materials acquire a magnetization parallel to $B$ [paramagnetism]
- Some materials acquire a magnetization opposite to $B$ [diamagnetism]
- Some materials retain their magnetization even after $B$ external is removed: ferromagnets

For those, the magnetization is not determined by the present field $B$, but by the whole $B$ history.
Example of ferromagnets: permanent magnets.

\[ \text{Torques & Forces on Magnetic Dipoles} \]

\[ \mathbf{m} \land \mathbf{m} = Iab \text{ magnet's dipole of a loop} \]

The loop inclined by \( \theta \)

\[ \mathbf{m} \] will experience torque

Looking side-wise

\[ \mathbf{F} \] and \( \theta i \)

Component forces that
Component on the horizontal are equal
In the vertical \( \theta \) generates a Torque:

\[ \mathbf{N} = a_i F \sin \theta \hat{x} \]

\[ F = I b \mathbf{B} \]
\[ \vec{N} = I \vec{b} \times \vec{B} \sin \theta \hat{x} = m \vec{B} \sin \theta \hat{x} \]

or

\[ \vec{N} = \vec{m} \times \vec{B} \]

torque on a current distribution for uniform B.

The torque is in the direction of aligning \( \vec{m} \) with \( \vec{B} \).

This is what accounts for paramagnetism.

The exclusion principle of Pauli dictates that electrons within a given atom lock together in pairs; what effectively pair of e's

\[ \uparrow \vec{m} \quad \downarrow \vec{m} \]

neutralizes the torque. As a result, paramagnetism normally occurs in atoms with an odd number of electrons, where the "extra" unpaired member is subject to the magnetic torque.

Alignment is never complete; random thermal collisions tend to destroy the order.

The net force on a current loop in uniform B is zero

\[ \vec{F} = I \oint (d\vec{l} \times \vec{B}) = I \left( \int \frac{d\vec{l}^2}{d\theta} \right) \times \vec{B}^3 = 0 \]

\[ = 0 \]

\( \downarrow \)

net displacement
Now, for a non-uniform field this is not the case.

\[ F = I \oint (d\hat{e} \times \mathbf{B}) \neq 0 \]

Example:

let say I suspend a circular wire of radius R

so \[ F = q \pi R I B \cos \theta \]

In general, for non-uniform B

\[ \vec{F} = \nabla (m \cdot \mathbf{B}) \]
Example: Find the force of attraction between two magnetic dipoles \( \mathbf{m}_1 \) and \( \mathbf{m}_2 \) separated by a distance \( r \).

\[
\mathbf{F} = \nabla (\mathbf{m}_2 \cdot \mathbf{B}) = \mathbf{m}_2 \cdot \nabla \mathbf{B}
\]

\(\mathbf{B}\) is due to \( \mathbf{m}_1 \) →

\[
\mathbf{B} = \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[ 3 \left( \mathbf{m}_1 \cdot \hat{\mathbf{r}} \right) \hat{\mathbf{r}} - \mathbf{m}_1 \right]
\]

\[
\mathbf{F} = \mathbf{m}_2 \cdot \frac{\partial}{\partial \mathbf{r}} \left\{ \frac{\mu_0}{4\pi} \frac{1}{r^3} \left[ 3 \left( \mathbf{m}_1 \cdot \hat{\mathbf{r}} \right) \hat{\mathbf{r}} - \mathbf{m}_1 \right] \right\}
\]

\[
= \frac{\mu_0}{4\pi} \frac{2 m_1 m_2 \hat{\mathbf{r}}}{r^4} \left( \frac{4}{r^3} \right) = \mu_0 m_1 m_2 \cdot \left( -\frac{3}{r^4} \right)
\]

\[
\mathbf{F} = -\frac{3\mu_0}{2\pi} \frac{m_1 m_2 \hat{\mathbf{r}}}{r^4} \quad \text{with } \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2
\]

**Effect of Magnetic Field on Atomic Orbits:**

Electrons not only spin, they also revolve around the nucleus.

Electrons spin and revolve around nucleus.
The orbit looks like a current:

\[ I = \frac{eV}{2\pi R} \]

The \textit{orbital} dipole moment: \( I_0 = I_0 R^2 = \frac{eV R^2}{2\pi} \)

\[ \hat{m} = -\frac{eV R \hat{z}}{2} \]

^negative\ charge\ of\ e^-\^

The \textit{orbital} magnetic dipole is subject to a torque \((\hat{m} \times \vec{B})\) too

but it's a lot harder to tilt the entire orbit \( \vec{B} \) than the spin—

so the contribution of the orbit is SMALL to paramagnetism.

However, there is another effect: speeding up or slowing down of the \( e^- \), depending on the orientation of \( \vec{B} \).

\[
\begin{align*}
\vec{B} & \quad \vec{B} \\
\Rightarrow & \quad \vec{B} \\
\vec{e}^- & \quad \vec{e}^+ \\
\rightarrow & \quad \rightarrow
\end{align*}
\]

Usually the centrifugal acceleration \( \frac{v^2}{R} \) is sustained by electric forces alone.

\[
\frac{1}{4\pi\varepsilon_0} \frac{e^2}{R} = \frac{m_e v^2}{R}
\]
In the presence of $\mathbf{B}$ there is an additional force:

$$-e(\mathbf{v} \times \mathbf{B})$$

so

$$\frac{1}{4\pi\varepsilon_0} \frac{e^2}{R} + e\mathbf{v} \cdot \mathbf{B} = \frac{m_e}{R} \mathbf{v}^2$$

so

$$e\mathbf{v} \cdot \mathbf{B} = \frac{m_e}{R} \mathbf{v}^2 - \frac{1}{4\pi\varepsilon_0} \frac{e^2}{R} - \frac{m_e}{R} \left(\mathbf{v}^2 - v^2\right) = \frac{m_e}{R} \left(\mathbf{v} + \mathbf{v}\right) \left(\mathbf{v} - \mathbf{v}\right)$$

Let's assume that $\Delta \mathbf{v} = \mathbf{v} - \mathbf{v}$ is small

So

$$e\mathbf{v} \cdot \mathbf{B} = \frac{m_e}{R} (\mathbf{v} + \mathbf{v}) \Delta \mathbf{v} \approx \frac{m_e}{R} (\Delta \mathbf{v} + 2\mathbf{v}) \Delta \mathbf{v}$$

$$\Delta \mathbf{v} = \frac{e\mathbf{v} \cdot \mathbf{B} R}{m_e (\mathbf{v} + \mathbf{v})} \frac{1}{e\mathbf{v} \cdot \mathbf{B} R} \frac{1}{m_e (\mathbf{v} + \mathbf{v})} \frac{e\mathbf{v} \cdot \mathbf{B} R}{m_e \mathbf{v} + \mathbf{v}} \approx e\mathbf{B} R \frac{1}{2 m_e}$$

So as $\mathbf{B}$ is turned on the electron speed up!

A change in $\mathbf{v}$ (orbital speed) change $\mathbf{m}$:

$$\Delta \mathbf{m} = -\frac{1}{2} e\mathbf{v} \cdot \mathbf{B} R \mathbf{a} = -\frac{e^2 R^2}{4 m_e}$$

So the change in $\mathbf{m}$ is the opposite of $\mathbf{B}^2$.

If the $-e^-$ will be orbiting opposite to the other way $\mathbf{m}$ will be pointing same direction to $\mathbf{B}$ and the $-e^-$ will be slowed down.
Magnetization

In the presence of $\mathbf{B}$ matter becomes magnetized. Two mechanisms:

1) paramagnetism (the dipoles associated with the spins of unpaired electrons experience a torque tending to line them up parallel to $\mathbf{B}$)

2) diamagnetism (the orbital speeds of e's is altered in such a way as to change dipole moment in a direction opposite to the field).

The state of magnetic polarization:

$$\mathbf{M} \equiv \text{magnetic dipole moment per unit volume}.$$