Now consider a closed loop.

\[ \oint \mathbf{V} \cdot d\mathbf{l} = 21.3 \mathbf{\hat{z}} - 24.0 = -2.67 \neq 0 \]

For certain fields \( \oint = 0 \).

**CLASS 3**
**SURFACE INTEGRAL**

Vector element of area \( da \)

\[ da = da \mathbf{\hat{n}} \]

Convention: direction of \( \mathbf{\hat{n}} \) \( \leftrightarrow \) right hand rule for open surface

If a closed surface: \( \mathbf{\hat{n}} \) outward
Surface integral

\[ \iiint_{\text{surface}} A \cdot d\mathbf{a} = \iint_{\text{surface}} A_x \, dx + A_y \, dy + A_z \, dz \]

Example: \( \mathbf{A} = yz \mathbf{x} + xe \mathbf{y} + xy \mathbf{z} \)

Surface: \( \frac{1}{4} \) circle in \( xy \) plane, radius \( a \)

\[ \hat{n} \text{ along } \mathbf{z} \quad d\mathbf{a} = dx \, dy \, \mathbf{z} \]

\[ y_{\text{max}} = \sqrt{a^2 - x^2} \]

\[ I = \iiint_{\text{surface}} (yz \mathbf{x} + xe \mathbf{y} + xy \mathbf{z}) \cdot (dx \, dy \, \mathbf{z}) \]

\[ I = \iint_{\text{surface}} \left[ xy \, dx \, dy \right] = \int_0^a \int_0^{\sqrt{a^2 - x^2}} xy \, dy \, dx = \ldots \]
Volume integral \[ I = \iiint V \, dV \] \[ = \iiint T \, dx \, dy \, dz \]

Example: \( T \) is density of a substance that might vary from point to point so \( I \) = total mass

\[ T = xy z^2 \]

\[ \iiint T \, dV = \iiint xy z^2 \, dx \, dy \, dz \]

\[ = \int_0^3 \int_0^1 \int_0^{1-y} x \, dx \, dy \, dz \]

---

Integral Theorems.

1. Green's Theorem

Divergence:

\[ \iint_S (\nabla \cdot V) \, dA = \iiint_V \nabla \cdot V \, dV \]

integral of the derivative over a volume = value of the function at the boundary (surface)
2. **Stokes Theorem**

\[ \oint_{\partial S} \mathbf{V} \cdot d\mathbf{a} = \iint_{S} \nabla \times \mathbf{V} \cdot d\mathbf{a} \]

\( \Rightarrow \) direction: right hand rule.

So if \( \mathbf{V} \times \mathbf{V} = 0 \) so \( \oint \mathbf{V} \cdot d\mathbf{a} = 0 \)

Curl measures "twist" of a vector.

\[ \nabla \times \mathbf{V} = \nabla \times \mathbf{V} \]

3. **Gradient Theorem**

\[ \int_{a}^{b} \nabla (T) \cdot d\mathbf{e} = T(b) - T(a) \]

depends only on the end points of a path (not on the path itself)

and \( \oint \nabla T \cdot d\mathbf{e} = 0 \)!
Example

\[ \mathbf{\hat{v}} = -\omega y \mathbf{\hat{x}} + \omega x \mathbf{\hat{y}} \]

\[ \oint (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint \mathbf{v} \cdot d\mathbf{l} \]

Let's take path of a circle with radius \( R \):

**RHS:**

\[ \mathbf{v} = \mathbf{dl} \]

\[ d\mathbf{dl} = k \, d\theta \]

\[ \mathbf{v} \cdot d\mathbf{l} = \omega k \, R \, d\theta \]

\[ \oint \mathbf{v} \cdot d\mathbf{l} = 2\pi \omega R^2 \]

**LHS:**

\[ \oint \mathbf{\nabla} \times \mathbf{v} = 2 \omega \mathbf{\hat{z}} \]

\[ \oint \mathbf{dl} \cdot \mathbf{\hat{z}} = 2\pi \omega R^2 = \text{RHS} \]
2nd derivatives

\[ \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \]

\[ \nabla^2 T = \nabla \cdot (T \nabla T) = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) T \]

This is because \[ \frac{\partial^2}{\partial x \partial y} = \frac{\partial^2}{\partial y \partial x} \]

\( x, y, z \) const in space + time

Not so in other coordinate systems
Spherical Coordinates

\[ X = r \sin \theta \cos \phi \]
\[ Y = r \sin \theta \sin \phi \]
\[ Z = r \cos \theta \]

math/physics azimuth

Astronomers' azimuth (local)

Applications

\[ \Theta = \text{colatitude} \]
\[ = 90^\circ - \text{latitude} \]
\[
\begin{align*}
\hat{r}, \hat{\theta}, \hat{\phi} & : \text{ unit vectors in direction of increasing } r, \theta, \phi \\
\text{Directions of } \hat{r}, \hat{\theta}, \hat{\phi} & \text{ change with } \hat{r}.
\end{align*}
\]

Unlike \( \hat{x}, \hat{y}, \hat{z} \)

1) Find \( \hat{r}, \hat{\theta}, \hat{\phi} \) in terms of \( x, y, z, \theta, \phi \)

Now \( \hat{r} = x \hat{x} + y \hat{y} + z \hat{z} \)

So \( \hat{r} = x \sin \theta \cos \phi + y \sin \theta \sin \phi + z \cos \theta \)

Generally \( \bar{A} = (\bar{A} \cdot \hat{x}) \hat{x} + (\bar{A} \cdot \hat{y}) \hat{y} + (\bar{A} \cdot \hat{z}) \hat{z} \)

\( \hat{\theta} = (\hat{\theta} \cdot \hat{x}) \hat{x} + (\hat{\theta} \cdot \hat{y}) \hat{y} + (\hat{\theta} \cdot \hat{z}) \hat{z} \)

Now \( \hat{\theta} \) is parallel to \( Z - \hat{r} \) plane.

\( \Theta \) \text{ is missing from the diagram.}

\( \Theta_{\text{hor}} = \cos \Theta \)

\( \Theta_{\text{vert}} = -\sin \Theta \)
\[
\mathbf{\phi} = \cos \theta \cos \phi \mathbf{x} + \cos \theta \sin \phi \mathbf{y} - \sin \theta \mathbf{z}
\]

\(\mathbf{\phi}\) is always parallel to the \(x-y\) plane.

\[
\mathbf{\phi} = -\sin \theta \mathbf{x} + \cos \theta \mathbf{y}
\]

Student should verify that

\[
\hat{\mathbf{r}} \cdot \mathbf{\phi} = \hat{\mathbf{\phi}} \cdot \mathbf{\phi} = 0
\]

Invert to get \(\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}\) in terms of \(\hat{\mathbf{r}}, \hat{\theta}, \hat{\phi}\).

2. Derivative - need to find \(\frac{\partial}{\partial r}\).

\[
\frac{\partial}{\partial r} = \hat{\mathbf{r}} \sin \theta \cos \phi + \hat{\mathbf{y}} \sin \theta \sin \phi + \hat{\mathbf{z}} \cos \theta
\]

\[
\frac{\partial}{\partial \theta} = -\hat{\mathbf{r}} \cos \theta \cos \phi + \hat{\mathbf{y}} \cos \theta \sin \phi - \hat{\mathbf{z}} \sin \theta.
\]

The unit vectors are functions of position and do not take them outside an integral.

Similarly,

\[
\frac{\partial}{\partial \phi} = \sin \theta \hat{\mathbf{\phi}}\frac{\partial}{\partial r} = 0
\]

\[
\frac{\partial}{\partial \theta} = -\hat{\mathbf{r}}\quad \frac{\partial}{\partial \phi} = \cos \theta \hat{\mathbf{\phi}}\quad \frac{\partial}{\partial r} = 0
\]
3) Expansion in unit vectors

\[
\vec{A} = A_x \hat{x} + A_y \hat{y} + A_z \hat{z} \\
= A_r \hat{r} + A_\theta \hat{\theta} + A_\phi \hat{\phi}
\]

Components: \( A_r = \vec{A} \cdot \hat{r} \), \( A_\theta = \vec{A} \cdot \hat{\theta} \), etc.

4) Find \( d\vec{r} \) (book: \( d\vec{r} \)) in spherical coordinates

\[
d\vec{r} = \left( \frac{\partial \vec{r}}{\partial r} \right) dr + \left( \frac{\partial \vec{r}}{\partial \theta} \right) d\theta + \left( \frac{\partial \vec{r}}{\partial \phi} \right) d\phi
\]

\[
\frac{\partial \vec{r}}{\partial r} = 2\hat{r} = \hat{r} + r\hat{\phi} = 0
\]

\[
\frac{\partial \vec{r}}{\partial \theta} = 2r \hat{\theta} = \frac{\partial r}{\partial \theta} \hat{r} + r \frac{\partial \hat{\theta}}{\partial \theta} = r \hat{\theta}
\]

\[
\frac{\partial \vec{r}}{\partial \phi} = r \frac{\partial \hat{\phi}}{\partial \phi} = r \sin \theta \hat{\phi}
\]

\[
d\vec{r} = dr \hat{r} + r d\theta \hat{\theta} + r \sin \theta d\phi \hat{\phi}
\]
Complication: \( \nabla \) operates on \( \bar{A} \), \( \bar{A}_\theta \), \( \bar{A}_\phi \), but also on \( \vec{r} \), \( \vec{\theta} \), \( \vec{\phi} \)

Result:

\[
\nabla \cdot \bar{A} = \frac{1}{r^2} \frac{2}{\partial r} (r^2 \bar{A}_r) + \frac{1}{rsin \theta} \frac{2}{\partial \theta} (sin \theta \bar{A}_\theta) \\
+ \frac{1}{rsin \theta} \frac{2}{\partial \phi} \bar{A}_\phi
\]

\[
\nabla \cdot \bar{A} = \frac{\partial}{\partial \theta} \left( \bar{A}_r \right) + \frac{\partial}{\partial \phi} \left( \bar{A}_\theta \right) + \frac{\partial}{\partial \phi} \left( \bar{A}_\phi \right)
\]

Use derivatives from topic (2).

\[ \nabla \times \bar{A}, \quad \nabla^2 \quad \text{in text,} \]

\( \nabla \times \bar{A} \) bal. \( \nabla^2 \) T book \( \nabla^2 \)
Volume \[ dV = r^2 \sin \theta \, dr \, d\theta \, d\phi \]

Area \[ d\vec{a} = r^2 \sin \theta \, d\theta \, d\phi \, \hat{r} \]

6) Gradient

Let \( T = T(r, \theta, \phi) \)

\[
\frac{dT}{dr} \, dr + \frac{\partial T}{\partial \theta} \, d\theta + \frac{\partial T}{\partial \phi} \, d\phi
\]

\[ = \vec{r} \cdot \nabla T \]

So \( \nabla T = \frac{\partial T}{\partial r} \hat{r} + \frac{\partial T}{\partial \theta} \hat{\theta} + \frac{\partial T}{\partial \phi} \hat{\phi} \)

because

\[
\vec{r} \cdot \nabla T = \frac{dr}{dr} \frac{\partial T}{\partial r} + \frac{d\theta}{d\theta} \frac{\partial T}{\partial \theta} + \frac{d\phi}{d\phi} \frac{\partial T}{\partial \phi}
\]

7) Divergence

\[
\nabla \cdot \vec{A} = \frac{\partial A_r}{\partial r} + \frac{1}{r} \frac{\partial A_{\theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial A_{\phi}}{\partial \phi}
\]

\[ A = A_r \hat{r} + A_{\theta} \hat{\theta} + A_{\phi} \hat{\phi} \]
**Fundamental Question:** What force on charge \( Q \) due to charges \( q_1, q_2, q_3 \)?

Coulomb's law: force on a point charge \( Q \) due to another charge \( q \):

\[
F = \frac{1}{4\pi \varepsilon_0} \frac{qQ}{r^2} \quad ; \quad r = \hat{r} - \hat{r}'
\]

This answer is based on experiments.

\( \varepsilon_0 \): permittivity of free space

In SI units:

\( \varepsilon_0 = 8.85 \times 10^{-12} \ \text{C}^2 / \text{N} \cdot \text{m}^2 \)

If \( q \) and \( Q \) have same sign: repulsive

If \( q \) and \( Q \) have opposite sign: attractive

For \( n \) charges:

\[
F_{\text{total on } Q} = \sum F_j
\]

**Electric Field:**

\[
E = \frac{\sum F_j}{Q} = \frac{1}{4\pi \varepsilon_0} \sum q \frac{1}{r^2}
\]

\( r = |\hat{r} - \hat{r}'| \)

\( \vec{E} \) is for several charges.

For continuous charge distribution:

\[
\vec{E} = \frac{1}{4\pi \varepsilon_0} \int \frac{\vec{q}}{r^2} \, d\vec{q}
\]

or

\[
d\vec{E} = \frac{dq}{4\pi \varepsilon_0} \frac{\hat{r} \cdot \hat{r}}{r^2}
\]

**Here we are invoking the principle of superposition:**

\[
\vec{E} = \vec{E}_1 + \vec{E}_2 + \vec{E}_3 + \vec{E}_4 + \ldots
\]

If, for example,

\[
F \propto (q_1 + q_2)^2
\]

Then \( F \neq F_1 + F_2 \)

\[
(q_1 + q_2)^2 \neq q_1^2 + q_2^2
\]

Superposition is not

A logical necessity but

An experimental fact.
For line: \( dq = \tau d\ell \) \( \Rightarrow \) constant charge per unit length

Surface: \( dq = \sigma d\ell \)

Volume: \( dq = \rho d\Sigma \)

\( \sigma(r) \) charge per unit area

\( \rho(r) \) charge per unit volume

Example: Uniform line charge, length 2L. What is \( E \) at \( z \) above midpoint?

Why Electrodynamics?

So why not yet write \( t \) and be done with it?

But \( E \) on \( A \) depends on separation distance \( r \), it depends both on their velocity \( v \) \& \( \theta \), and on the on the acceleration of \( q \).

Moreover it is not the position, velocity \& acceleration of \( q \) that matters.

Electromagnetic waves travel with speed of light

so what concerns \( \Theta \) is the position, velocity, and acceleration of \( q \) had at early times when the message left.

So to begin with, we will start with Electrodynamics \( \Rightarrow \) the source terms are stationary.