- Last class we saw work & energy
- How are you studying? Next class HW!

**Today**

Conductors: contain many "free charges"

Metallic conductors: contain 1 or more e⁻'s free to roam

Why do the e⁻'s not escape?

If an e⁻ tries to leave the surface, it feels an attractive force → work must be done to leave (~ few eV)

**Theorem**

\[ \vec{E} \] inside a conductor = 0

If \( \vec{E} \neq 0 \), charges will move until \( \vec{E} = 0 \).

If you put a conductor in an external field \( \vec{E}_0 \)

\[ \begin{array}{c}
- \\
- \\
- \\
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\end{array} \]

\[ \begin{array}{c}
- \\
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\end{array} \]

\[ \vec{E}_0 \]

\[ \vec{E}_1 \] is created in the opposite direction to \( \vec{E}_0 \) in order to cancel \( \vec{E}_0 \)
It's only the $\Theta$ charges that move.

**Theorem**  $\rho = 0$ inside a conductor

$$\nabla \cdot \vec{E} = \frac{\rho}{\epsilon_0}$$

If $\vec{E} = 0 \rightarrow \rho = 0$

There is still a charge, but the net charge is zero.

**Theorem**  A conductor is an equipotential

$$V(b) - V(a) = -\int_a^b \vec{E} \cdot d\ell$$

If $\vec{E} = 0 \int_a^b \vec{E} \cdot d\ell = 0$ so $V(b) = V(a)$

**Theorem**  $\vec{E}$ is normal to the surface of a conductor

Otherwise the charge will flow around the surface until it kills off...
the tangential component.

Induced charges

\[ +q \]

The charges in the conductor move such that the field inside the conductor is zero.

Since the negative charge is closer to \(+q\), there is a net force of attraction. What if there is a "cavity" inside the conductor? Inside the cavity, \(+q\)

\[ E \neq 0 \]

No external fields penetrate the conductor \(\Rightarrow\) they are canceled at the outer surface by the induced charges.
Let's use a Gauss surface:

\[ \oint E \cdot dA = 0 \quad \text{so} \quad \Phi_{\text{enc}} = q + q_{\text{ind}} \Rightarrow q_{\text{indwud}} = -q \]

no \( E \) in conductor!

The charge on the outer surface of the conductor is \( +q \); the charge "communicates" the presence of the cavity \( +q \) to the outside world!

Example:

An uncharged spherical conductor centered at the origin has a cavity of some wiered shape. Inside the cavity there is a charge \( q \). What is \( E \) outside the sphere?
so, outside the conductor \[ \mathbf{E}(P) = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2} \]

\[ \mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0} \mathbf{\hat{n}} \]

\[ \oint \mathbf{E} \cdot d\mathbf{a} = \frac{\sigma A}{\epsilon_0} = \mathbf{E}_{\text{above}} \mathbf{A} - \mathbf{E}_{\text{below}} \mathbf{A} = \frac{\sigma A}{\epsilon_0} \]

\[ \mathbf{E}_{\text{above}} - \mathbf{E}_{\text{below}} = \frac{\sigma}{\epsilon_0} \mathbf{\hat{n}} \]

For a surface of a conductor, \( \mathbf{E}_{\text{below}} \perp = 0 \)

So \[ \mathbf{E}_{\text{above}} \perp = \mathbf{E}_{\text{above, total}} = \frac{\sigma}{\epsilon_0} \mathbf{\hat{n}} \]
With \( \nabla V = -\vec{E} \)

\[
\frac{\partial V}{\partial n} = -\frac{\sigma}{\varepsilon_0} \quad \text{or} \quad \sigma = -\varepsilon_0 \frac{\partial V}{\partial n}
\]

Force per unit area on the surface charge \( \sigma \) is:

\[
\vec{F} = \sigma \vec{E}
\]

\[
\vec{F} = \frac{\sigma^2}{\varepsilon_0} \hat{n} \cdot \left( \frac{1}{2} \right)
\]

comes from averaging the field from below and above.

\( \text{Capacitors} \)

\( \text{Potential} \)

\( V \) is constant over a conductor so the potential between them is

\[
V = V_+ - V_- = -\int_{-}^{+} \vec{E} \cdot d\vec{l}
\]
We don't know $E$. But we know since

$$E^g = \frac{1}{4\pi \varepsilon_0} \int \frac{\rho}{r^2} \, d\sigma$$

that $E \propto \mathcal{Q}$ (charge).

So if $2 \times \rho \rightarrow 2 \times E$

We define then

$$C = \frac{\mathcal{Q}}{V}$$

capacitance

$$[C] = \text{Faraday (F)}$$

$V$ by definition is the potential of a positive conductor less than the negative one $\mathcal{Q}$.

is the charge $\mathcal{Q}$.

So $C > 0$

If we talk about a capacitance of a single conductor, the second conductor with $\mathcal{Q}$ charge is an imaginary spherical shell with $r \to \infty$, so it doesn't contribute.

Example

parallel plate conductor

What is the capacitance?
Let's put \( +Q \) on the conductor above and \( -Q \) on the bottom.

(I) \[ \text{ +Q } \]

(II) \[ \text{ -Q } \]

\[ E_+ \]

\[ E_- \]

\[ E_+ \]

\[ E_- \]

Corresponding to \( \sigma = \frac{Q}{A} \)

In region II we will have \( 2\mathcal{E} \).

Each plate \( \ell \) produces \( \sigma = \frac{Q}{2\mathcal{E}_0} \).

The total field \( E_+ = \frac{Q}{\mathcal{E}_0 A} \) and \( V = \frac{Q \ell}{\mathcal{E}_0 A} \) \( \Rightarrow \) \( \zeta = \frac{Q}{V} = \frac{Q \mathcal{E}_0 A}{Q \ell} = \frac{\varepsilon A}{d} \)

Example: Two concentric spherical metal shells, with radii \( a \) and \( b \).

What is \( \zeta \) ?

Let's put \( +Q \) and \( -Q \).
The field between them is

\[ \vec{E} = \frac{1}{4\pi\varepsilon_0} \frac{Q}{r^2} \hat{r} \]

and

\[ V = -\int_a^b \vec{E} \cdot d\vec{l} = -\frac{Q}{4\pi\varepsilon_0} \int_a^b \frac{1}{r^2} dr = \frac{Q}{4\pi\varepsilon_0} \left( \frac{1}{a} - \frac{1}{b} \right) \]

So

\[ \zeta = \frac{Q}{V} = \frac{4\pi\varepsilon_0}{b-a} \]

What is the work needed to "charge up" a capacitor? (to a final \( Q \))

\[ dW = \left( \frac{q}{C} \right) dq \]

\[ \text{since } W = Q \left( V(b) - V(a) \right) = Q \int \frac{q}{C} dq = Q \frac{q^2}{2C} \]

\[ W = \int_{Q=0}^{Q} \frac{q^2}{2C} dq = \frac{Q^2}{2C} = \frac{C V^2}{2C} \]

\[ W = \frac{V^2 C}{2} \]
Now we enter in Chapter 3 (do the problems for HW)

Let's examine some special techniques:

- Laplace Equation

Usually we need to find \( E \); it's hard usually

\[
E^2 = \frac{1}{4\pi \epsilon_0} \int \frac{\rho(r')}{r^2} \, d\sigma'
\]

Coulomb law

since the charge distribution might be complicated. So we use potential

\[
V(r) = \frac{1}{4\pi \epsilon_0} \int \frac{\rho(r')}{r} \, d\sigma'
\]

but this can also be hard. Also in problems involving conductors
the only thing we know are total charge or potential.

So let's cast the problem is a Laplace Equation. What it is?

\[ \nabla^2 V = -\frac{\rho}{\epsilon_0} \]

If we are in a region where \( \rho = 0 \) we get the Laplace Equation

\[ \nabla^2 V = 0 \]
That in Cartesian coordinates is

\[
\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0
\]

In 1-D \( \frac{d^2 V}{dx^2} = 0 \) \( \therefore V(x) = mx + b \)

\( m \), \( b \)?

Example: Suppose \( V(x=1) = 4 \) and \( V(x=5) = 0 \) so

\[
\frac{dV}{dx} = m \quad \rightarrow \quad \frac{\Delta V}{\Delta x} = m \quad \rightarrow \quad m = -1
\]

and \( b = 5 \)

\( V(x) = -x + 5 \) or

Some properties of this solution:

- \( V(x) \) is an average of \( V(x+a) \) \& \( V(x-a) \) for any \( a \)

\[
V(x) = \frac{1}{2} [V(x+a) + V(x-a)]
\]

\[
= \frac{1}{2} m(x+a) + \frac{1}{2} b + \frac{1}{2} m(x-a) + \frac{1}{2} b = mx + b
\]
In this sense $V(x)$ is an average of the values of the left and to the right of $x$.

- Laplace equation do not tolerate any maxima or minima. I.e.: extreme values of $V$ occur at the end points.