Diffraction

What is it? Any deviation from geometrical optics that results from
obstruction of a wavefront by an object. A universal wave phenomenon.
Examples in everyday life: CD or DVD rainbow, credit card hologram, ring around moon,...

When a wave encounters an obstacle → diffraction occurs
→ region of wavefront
is altered in amplitude &
phase.

[Note: No difference in physical mechanism between interference & diffraction]

In order to better understand diffraction → Hugen's Principle

→ Every pt. on a wavefront of light can be viewed as the
source of secondary spherical wavelets. And, Frendel added,
the field at any pt. after = superposition of all these wavelets
(taking into account amplitudes and phases)

wavelets, have same freq.
of primary wave.

[Diagram: Incident plane wave, hole creates a pt. source, spherical wavelets spreading in direction.]

So, for example, in calculating the diffraction pattern
from a double-slit at some point on a screen,
one would consider every pt. off the wavefront emerging from
each slit as a source of secondary wavelets →
Superposition of these produce the resultant field.

In add up amplitudes & take into account phase...
Diffraction is just an extension of our discussion on interference. To understand diffraction, let us consider interference arising from \(N\) sources, which we assume for simplicity are identical.

\[ r_1, r_2, \ldots = \text{distance to observation point, } P \]

(far away)

Assume: no initial phase difference.

- Sources are small compared to \(\lambda\).

The amplitudes of separate waves arriving at \(P\) are all identical:

\[ E_0(r_1) = E_0(r_2) = E_0(r_3) = \ldots E_0(r_n) = E_0(r) \]

-if the spatial extent of array is small compared to distance \(P\).

The sum of interfering wavelets at \(P\) is the

real part of:

\[ E = E_0(r)e^{-i\omega t} + E_0(r)e^{-i\omega t} + \ldots E_0(r)e^{-i\omega t} \]

this can be re-arranged to:

\[ E = E_0(r)e^{-i\omega t} \sum_i \left( k(r_i - r) = k(r_2 - r) + k(r_3 - r) + \ldots + k(r_n - r) \right) \]

from figure above, can see that \(r_2 - r = dsin\theta\)

and the phase difference between adjacent sources is

\[ S = k_0 \delta \]

\[ \Delta \text{ path difference} \]

So we can write the field at point \(P\) as:

\[ E = E_0(r)e^{-i\omega t} ikr \left[ 1 + e^{i8} + (e^{i8})^2 + \ldots (e^{i8})^{n-1} \right] \]

this geometric sum has the solution:

\[ \frac{e^{i8} - 1}{e^{i8} - 1} = \frac{e^{i8/2} - e^{-i8/2}}{e^{i8/2} - e^{-i8/2}} \]
\[
\begin{align*}
\text{Distance} \quad & \quad \text{Area} \\
& = \text{Distance} \cdot \text{Area} \quad \text{Volume} \\
& = \text{Distance} \cdot \text{Area} \cdot \text{Volume} \\
\end{align*}
\]
Now, an infinitesimal segment of line source, \(dy\) will contain:

\[
E_i = \frac{E_0}{L} \sin(kr_i - wt) \left(\frac{N\Delta y}{D}\right)
\]

Let us imagine that array is divided up into \(M\) such segments.

Then the contribution to the E-field at point \(P\) will be

\[
E_i = \frac{E_0}{L} \sin(kr_i - wt) \left(\frac{N\Delta y_i}{D}\right)
\]

(Note we assume \(\Delta y\) is small enough that there is a negligible relative phase shift within each segment.)

To make this a continuous line source, let \(N \rightarrow \infty, \Delta y \rightarrow dy\)

But total energy output must be finite so each \(E_i \rightarrow 0\)

So we define source strength per unit length:

\[
E_L = \frac{1}{L} \lim_{N \rightarrow \infty} \left(\frac{N E_0}{D}\right)
\]

Then, net field at pt. \(P\) from all segments is

\[
E = \sum_{i=1}^{M} E_i = \frac{E_L}{L} \sin(kr - wt) \Delta y
\]

To go from discrete \(\rightarrow\) continuous, we integrate across entire length of line:

\[
E = E_L \int_{-D/2}^{D/2} \sin(wt - w\phi) \, dy
\]

\(\phi = \text{angle from line element to point } P\).
Fraunhofer Diffraction

Consider case \( R >> D \), then \( r(y) \approx R \), so can take out of integral:

\[
E = \frac{E_0}{R} \int_{-D/2}^{D/2} \sin \left( \frac{\pi r(y)}{l} \right) dy
\]

Let no impulse just constant is given.

Later using Cauchy's integral formula.

Let us consider unit circle.

Using law of cosines:

\[
c^2 = a^2 + b^2 - 2ab \cos \theta
\]

we get:

\[
r(y) = y^2 + R^2 y R \cos(90^\circ - \theta)
\]

\[
= y^2 + R^2 - 2yR \sin \theta
\]

Doing a Taylor expansion about \( y = 0 \):

\[
[f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \ldots]
\]

\[
f(y) = r(y) = (y^2 + R^2 - 2yR \sin \theta)^{1/2}
\]

Here

\[
f(0) = R
\]

\[
f'(y) = \frac{1}{2} (R^2 + y^2 - 2yR \sin \theta)^{-1/2} (2y - 2R \sin \theta)
\]

\[
f'(0) = \frac{1}{2} (R^2)^{-1/2} (-2R \sin \theta) = -\sin \theta
\]

So we get:

\[
r(y) = R - y \sin \theta + \frac{y^2}{2R} \cos^2 \theta + \ldots
\]
Consider the case when $R$ is large.

\[ r(y) = R - y \sin \theta + \frac{y^2}{2R} \cos^2 \theta + \ldots \]

- Can ignore this and higher order terms as long as $R$ is large enough that the $\frac{1}{R}$ term in the $\sin(2\pi \cdot \frac{y}{R} \cdot \omega t)$ is negligible.

- So even when $y = \pm \frac{D}{2}$ (max value), we require $\frac{kr}{2}$ the third term in $r$ to be small or

\[ kR \ll \frac{\pi \sqrt{D^2}}{4 \pi R} \Rightarrow \frac{\pi D^2}{4 \pi R} \ll \frac{R^2}{4 \pi R} \Rightarrow \frac{\pi D^2}{4 \pi R} \ll \frac{R}{4 \pi R} \]

- This will be small as long as $R$ is large enough.

This approximation is called the **Fraunhofer condition**.

Substituting for $r$ in integral yields:

\[ E = \frac{\varepsilon L}{R} \int_{-D/2}^{D/2} \sin \left( kR \left( R - y \sin \theta \right) - \omega t \right) \, dy \]

which can be integrated to get:

\[ E = \frac{\varepsilon L D}{R \sin \theta} \sin \left( \frac{kD}{2} \sin \theta \right) \sin \left( kr \omega t \right) \]

To simplify, we introduce:

\[ \beta = \frac{kD}{\sin \theta} \]

Then,

\[ E = \frac{\varepsilon L D}{R} \left( \frac{\sin \beta}{\beta} \right) \sin \left( kr \omega t \right) \]
So, the intensity (typical observable):

\[ I(\theta) \propto \langle E^2 \rangle \]

\[ = \left( \frac{E \cdot D}{R} \right)^2 \left( \frac{\sin^2 \beta}{\beta^2} \right) \langle \sin^2 (kR \cdot \cos \theta) \rangle \]

\[ \propto \frac{E \cdot D}{R} \left( \frac{\sin^2 \beta}{\beta^2} \right)^{1/2} \]

If \( \theta = 0 \) \( \frac{\sin \beta}{\beta} \approx 1 \) \( \rightarrow I(\theta) = \text{max} \) \( \rightarrow \text{principle maximum} \)

\[ I(\theta) = I(0) \cdot \left( \frac{\sin \beta}{\beta} \right)^2 \]

Since \( \beta = \frac{\pi D \sin \theta}{\lambda} \); when \( D >> \lambda \) \( \rightarrow I(\theta) \) drops rapidly

when it deviates from \( \theta = 0 \)

Then from the expression for \( E \):

\[ E = \frac{E \cdot D}{R} \left( \frac{\sin \beta}{\beta} \right) \sin (kR \cdot \cos \theta) \]

\( \text{drops drastically for } \theta \neq 0 \)

can think of the line source as a single point emitter a distance \( R \) from point \( P \), radiating along \( \theta = 0^\circ \) direction.

On the other hand, if \( \lambda >> D \) \( \rightarrow \beta = \frac{\pi D \sin \theta}{\lambda} \) small

\( \rightarrow \sin \beta \approx \beta \) \( \rightarrow I(\theta) \approx I(0) \) \( \rightarrow \text{constant for all } \theta \)

\( \rightarrow \) so line source resembles a point source emitting spherical waves.