Overview:

EM waves when they encounter materials create complex interactions with charged particles in the medium.

Charges oscillate $\rightarrow$ radiate $\rightarrow$ net field $=$ Superposition of both source & emitted waves

We have already seen

These complicated effects can for most purposes be described by parameters: index of refraction $\mu$ absorption coefficient $\kappa$ of the material. We already studied these parameters for conducting materials. Here we study it for dielectrics or non-conducting materials (i.e., no free $e^-$)

Dielectric medium:

For a dielectric, $\sigma=0$ and Maxwell's equations give:

$$\nabla^2 E = \mu_0 \frac{\partial^2 E}{\partial t^2}$$

Let us assume $\kappa_m = 1$ so $\mu = \mu_0$ ($\kappa_m = \frac{\mu}{\mu_0}$)

Recall that $\varepsilon E = D = \varepsilon_0 E + P$ (effect of field in a dielectric medium $\rightarrow$ induced dipole)

Then $\nabla^2 \varepsilon E = \mu_0 \frac{\partial^2 D}{\partial t^2} = \mu_0 \frac{\partial^2 (\varepsilon E + \frac{P}{\varepsilon_0})}{\partial t^2}$

$\frac{\partial^2 \varepsilon}{\partial t^2} = \frac{\partial^2 E}{\partial t^2}$

Then $\frac{\partial^2 P}{\partial t^2} = \mu_0 \frac{\partial^2 E}{\partial t^2}$

$\vec{P} =$ Polarization field = dipole moment/unit volume

Static case: For bound $e^-$ with $N = \#$/unit volume, $\text{charge} = -e$,

dispaced in response to external field $\vec{E}$ by $\vec{x}$

Then $\vec{P} = -N \vec{x} e^-$

(If $\vec{E}$ static)
For equivalent spring constant:

\[ F = -\varepsilon E = kx \]

\[ \frac{\partial}{\partial \rho} = \frac{N e^2}{\varepsilon} \]

(more massive nuclei can be considered static and they cannot respond fast enough to the EM field.)

Electrons behave as though the forces binding them to the nuclei are elastic forces given by Hooke's Law, where the restoring force is a displacement and acts in opposite direction.

**Dynamic Case**

In an alternating E field, forced oscillations remove some energy from incident radiation. A model of oscillating e⁻ can be described by a damped harmonic oscillator:

\[ m \frac{d^2x}{dt^2} = F_{\text{spring}} + F_{\text{damping}} + F_{\text{driven}}. \]

\[ \Rightarrow m \frac{d^2x}{dt^2} + m \gamma \frac{dx}{dt} + kx = -\varepsilon E_0 e^{-i\omega t} \]

\[ \frac{d^2x}{dt^2} + \gamma \frac{dx}{dt} + \frac{k}{m} x = -\frac{\varepsilon}{m} E_0 e^{-i\omega t} \]

Set \( x = x_0 e^{-i\omega t} \), then

\[ \frac{dx}{dt} = -i\omega x_0 e^{-i\omega t} \]

\[ \frac{d^2x}{dt^2} = -\omega^2 x_0 e^{-i\omega t} \]

So \( e^{-\gamma t} \) becomes:

\[ -\omega^2 x_0 - i\gamma \omega x_0 + \omega_0^2 x_0 = -\frac{\varepsilon}{m} E_0 \]

\[ \Rightarrow x_0 = \frac{-\varepsilon m E_0}{\omega_0^2 - \omega^2 - i\gamma \omega} \]

The polarization is then:

\[ \rho = -Ne^2 \left( \frac{1}{\omega_0^2 - \omega^2 - i\gamma \omega} \right) \frac{E}{m} \]
Using Eq. (1), we have:
\[
\nabla^2 E = \mu_0 \varepsilon_0 \frac{\partial^2 E}{\partial t^2} \left[ 1 + \frac{Ne^2}{m\varepsilon_0} \left( \frac{1}{\omega_0^2 - \omega^2 - i\kappa \omega} \right) \right]
\]

which has solution:
\[
E = E_0 e^{i(kz - \omega t)}
\]

which means:
\[
-k^2 E = -\mu_0 \varepsilon_0 \omega^2 \left( 1 + \frac{Ne^2}{m\varepsilon_0} \left( \frac{1}{\omega_0^2 - \omega^2 - i\kappa \omega} \right) \right) E
\]

\[
k^2 = \frac{\omega^2}{c^2} \left( 1 + \frac{Ne^2}{m\varepsilon_0} \left( \frac{1}{\omega_0^2 - \omega^2 - i\kappa \omega} \right) \right)
\]

so \( k \) is complex.

Now \( n = \frac{c}{\omega} = \frac{c}{\omega_0} = \frac{k\varepsilon_0}{c} \Rightarrow n^2 = \frac{c^2}{\omega^2} k^2 \)

so \( n^2 = 1 + \frac{Ne^2}{m\varepsilon_0} \left( \frac{1}{\omega_0^2 - \omega^2 - i\kappa \omega} \right) \)

(complex \( n \), \( n = \text{Re}(n) + i\text{Im}(n) \))

\( \text{Im}(n) = \text{extinction index} \)

Since \( k = \frac{\omega}{c} \); \( i(kz - \omega t) \)

and \( E = E_0 e^{i(kz - \omega t)} \)

\[
E = E_0 e^{-\omega \kappa \text{Im}(n) z} e^{i(\omega \kappa \text{Re}(n) z - \omega t)}
\]

\[
= E_0 e^{-\omega \kappa \text{Im}(n) z}. E
\]
Plot of $n_I$ and $n_E$ vs. $\omega$ looks like:

Above $\omega_0$, damping large.

Below $\omega_0$, Re($\gamma$) increases with $\omega$.

$\Rightarrow$ "normal dispersion"

Across resonance, Re($\gamma$) decreases with $\omega$.

$\Rightarrow$ "anomalous dispersion"